The effect of exchange rates on (Statistical) decisions

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Teddy Seidenfeld Mark J. Schervish Joseph B. (Jay) Kadane

Carnegie Mellon University

Part 1: What do fair prices reveal about subjective probabilities?

Part 2: What does elicitation by a (strictly) proper scoring rule reveal?

Part 1: Introduction – subjective probability as fair betting rates.

- **Consider bets where the stakes are monetary.**

Assume that Smith's preferences over bets,

when formulated in dollars and with modest stakes,

satisfy (de Finetti's) structural assumptions for fair-prices – *previsions*.

Here are the formal details of de Finetti's prevision theory.

Let $\chi = \{X_i: S \rightarrow \Re;\ i = 1, \ldots\}$ be variables measureable w.r.t algebra $\mathcal{B}$ over S.

(Unconditional) COHERENCE$_1$ as *fair-prices*

The 2-person, 0-sum sequential *prevision* game

Bookie **moves first and sets a fair (buy/sell) price $P(X)$ for each $X \in \chi$,**

The *Gambler* then acts on the *Bookie*'s offers. The *Gambler* – may make *finitely many* (non-trivial) contracts at the *Bookie*'s announced prices.

For finitely many X, *Gambler* fixes a non-zero real number, β_X, which determines a contract.

In state s, a contract has an *outcome* to the *Bookie* (with negative outcome to the *Gambler*) of $\beta_X[X(s) - P(X)]$.

The *Bookie*'s net *outcome* in state ω is the sum of the payoffs from the finitely many non-zero contracts: $\sum_{X \in \chi} \beta_X[X(s) - P(X)]$.

***Coherence$_1$*: The *Bookie*'s previsions are *incoherent$_1$* if there is an acceptable *finite* combination of gambles with *uniformly negative* net-payoff.**

Otherwise the previsions are *coherent$_1$*.

"Book" Theorem **(de Finetti, 1937):**

A set of previsions {*P*(*X*)} are *coherent*$_1$

<u>if and only if</u>

There exists a (finitely additive) probability *P* on Ω such that the previsions are the *P*-Expected values of the corresponding variables $\mathcal{E}_P[X] = P(X)$.

The Dollar-Yen puzzle

For simplicity, consider two states of interest: s_1 and s_2.

Bet B **pays \$1.00 if s_1 obtains, and -\$1.00 if s_2 obtains.**

Suppose Smith finds Bet B is *fair*; hence,

$$P^{\$}(s_1) = P^{\$}(s_2) = 1/2.$$

In detail, let $I_1(\omega)$ be the indicator function for state ω_1:

$$I_1(s) = 1 \text{ if } s = s_1$$

$$I_1(s) = 0 \text{ if } s = s_2$$

***Bet B in US dollars*:**

$$\beta[I_1(s) - P^{\$}(s_1)]$$

$$= \quad 2[I_1(s) - 1/2]$$

Next, we offer Smith bets on the same two states but this time with monetary payoffs in Yen.

Again, we suppose Smith's preferences for bets in Yen satisfy de Finetti's structural assumptions – she/he has *previsions* for contracts in Yen

Bet B' pays 100¥ if s_1 obtains, and -125¥ if s_2 obtains.

Smith finds bet B' is *fair*; hence,

$$P^{¥}(s_1) = 5/9 \quad \text{and} \quad P^{¥}(s_2) = 4/9.$$

***Bet B' in Japanese Yen*:** $\beta[I_1(s) - P^{¥}(s_1)]$

$$= 225[I_1(s) - 5/9]$$

***Question*: As $P^{\$}(s_i) \neq P^{¥}(s_i)$, are Smith's combined previsions incoherent?**

***Answer*:** **NO!**

Let the states indicate the rate of exchange between the two currencies:

- **in state s_1, \$1 ≈ 100¥;**
- **in state s_2, \$1 ≈ 125¥;**

Then *Bet B* is equivalent to *Bet B'*.

The one bet is *fair* if and only if the other is.

Resolution of the puzzle

Decisions (acts) as functions from states to outcomes

The canonical decision matrix: **decisions × states**

	s_1	s_2			s_j			s_n
d_1	O_{11}	O_{12}			O_{1j}			O_{1n}
d_2	O_{21}	O_{22}			O_{2j}			O_{2n}
d_k	O_{k1}	O_{k2}			O_{kj}			O_{kn}

$d_i(s_j)$ = outcome $\mathbf{o_{ij}}$.

What are "outcomes"?

That depends upon which version of expected utility. Allow arbitrary outcomes,

providing that they admit a von Neumann-Morgenstern cardinal utility *U*(•).

A central theme of Subjective Expected Utility [*SEU*] is this:

- **axiomatize (weak) preference $\preceq$ over decisions so that**

$$d_1 \preceq d_2 \ \ iff \ \ \Sigma_j P(s_j)U(o_{1j}) \leq \Sigma_j P(s_j)U(o_{2j}),$$

for one subjective (personal) probability P(•) defined over *states*

and one cardinal utility U(•) defined over *outcomes*.

- **Then the decision rule is to choose that (an) option that *maximizes SEU*.**
- **The Representation theorem promises a *unique* decomposition into a pair {P, U} where P is the agent's subjective probability over the states and U is her/his cardinal (state-independent) utility for outcomes.**

Note: In this version of SEU:

(1) decisions and states are probabilistically independent, $P(s_j) = P(s_j \mid d_i)$.

Reminder: This is sufficient for a fully general *dominance* principle.

(2) Utility is state-independent, $U_j(o_{ij}) = U_h(o_{gh})$, if $o_{ij} = o_{gh}$.

Here, $U_j(o_{\bullet j})$ is the conditional utility for outcomes, given state s_j.

(3) (Cardinal) Utility is defined up to positive linear transformations,

$U'(\bullet) = aU(\bullet) + b$ (a > 0) is also the same utility function for purposes of *SEU*.

Note: Under these circumstances with act/state prob. independence, utility is defined up to a similarity transformation: $U_j'(\bullet) = aU_j(\bullet) + b_j$.
So, maximizing SEU and Maximizing Subjective Expected Regret-Utility are equivalent decision rules.

On the structural assumptions for the Representation Theorem

- ***Act-state independence*: no cases of "moral hazards" are considered – so strict dominance is valid.**

Reminder: **Consider the following binary state, two act decision problem, with outcomes ordinally (or cardinally) ranked so that more is better.**

	ω_1	ω_2
Act_1	**3**	**1**
Act_2	**4**	**2**

Act_2 strictly dominates Act_1. Nonetheless, if

$$Prob(\omega_i \mid Act_i) \approx 1 \quad (i = 1, 2),$$

then dominance carries no force. A rational agent prefers Act_1 to Act_2.

- ***State-independent utility*: no cases where the value of a prize depends upon the state in which it is received.**

Reminder: **Once we entertain, generalized state-dependent utilities for prizes, there is maximal under-determination (= up to *mutual absolute continuity* of probability) of probability/utility pairs that represent the very same preference ranking of acts. Then, elicitation is hopeless!**

Matrix of *m*-many acts on the partition of *n*-many uncertain states

	ω_1	ω_2		ω_j			ω_n
Act_1	o_{11}	o_{12}		o_{1j}			o_{1n}
Act_2	o_{21}	o_2		o_{2j}			o_{2n}
Act_i	o_{i1}	o_{i2}		o_{ij}			o_{in}
Act_m	o_{m1}	o_m		o_{mj}			o_{mn}

In accord with *generalized* (possibly state dependent) SEU preferences:

Act_1 is dispreferred to Act_2 *if and only if* $\sum_j P(s_j)U_j(o_{1j}) \leq \sum_j P(s_j)U_j(o_{2j})$.

- **The probability P has no subscript – no moral hazard.**
- **Utility U has a subscript for states: possible state-dependent utility.**

Choose P* mutually absolutely continuous with P and define the constants

$$c_j = P(s_j)/P^*(s_j)$$

and let $U^*_j(\bullet) = c_j U_j(\bullet) \qquad (j = 1, \ldots, n).$

Then, trivially,

$$\sum_j P(s_j)U_j(o_{1j}) \leq \sum_j P(s_j)U_j(o_{2j})$$

if and only if $\sum_j P^*(s_j)U^*_j(o_{1j}) \leq \sum_j P^*(s_j)U^*_j(o_{2j}).$

- **So, in the presence of state-dependent utilities, elicitation of the decision maker's degrees of belief using only her/his preferences over acts is entirely impossible!**
- **Note well: This impossibility arises even when one of the representations is by a state-independent utility!**

The Dollar-Yen example – *<u>Diagnosis</u>*

We have constructed a partition with two states that precludes a *state-independent* utility over *<u>both</u>* US $ and Japanese Yen ¥ *simultaneously*.

With the deFinetti "*Book*" theorem in $-bets, Smith constructs a state-independent utility for dollars that is state-dependent for Yen.

Similarly, when we use the ¥-bets, Smith constructs a state-independent utility for Yen that is state-dependent for US dollars.

- **The relations between these to pairs $\{P^{\$}, U_{\$}\}$ and $\{P^{¥}, U_{¥}\}$ is that each successfully represents Smith's preferences. Each is "correct."**

***Principal Claim*:**

> ***Whether or not Smith holds a state-independent utility for Yen or for Dollars is <u>underdetermined</u> by the entirety of his/her betting behavior!***

***Part 2*: Lessons for elicitation with (strictly) proper scoring rules.**

- **De Finetti was fully aware that the *prevision-game* admits strategic play on the part of the Bookie, who might have opinions about the Gambler's personal previsions for the same events.**

COHERENCE$_2$ & COHERENCE$_{2+}$

***Coherence$_2$* for the one-person *Forecasting Game* with Brier Score:**

There is only the one player in the *Forecasting Game*,
the *Forecaster*, who announces a forecast $F(X)$ for each X in χ.

In state s, the *Forecaster* is penalized $-[X(s) - F(X)]^2$

The *Forecaster*'s net score in state ω from forecasting finitely variables $\{F(X_i): i = 1, \ldots, k\}$ is the sum of the k-many individual losses: $\sum_{X \in \chi} -[X_i(s) - F(X_i)]^2$.

***Coherence*$_2$**: **The *Forecaster*'s forecasts $\{F(X): X \in \chi\}$ are *coherent*$_2$ provided that there is no *finite set* of variables, $\{X_1, \ldots, X_k\}$ and set of rival forecasts $\{F'(X_1), \ldots, F'(X_k)\}$ that yields a uniformly smaller net loss for the *Forecaster* in each state.**

$$\neg \exists (\{F'(X_1), \ldots, F'(X_k)\}, \varepsilon > 0), \forall \omega \in \Omega$$

$$\sum_{X \in \chi} [X_i(s) - F'(X_i)]^2 \leq \sum_{X \in \chi} [X_i(s) - F(X_i)]^2 - \varepsilon.$$

Otherwise, the *Forecaster*'s forecasts are *incoherent*$_2$.

***Elicitation**: Brier Score* **is *strictly* proper – an SEU decision maker with personal probability *P* on *S* (uniquely) maximizes her/his SEU by forecasting her/his subjective expected value of *X*: $F(X) = \mathcal{E}_P(X)$.**

The general form of scoring rule that we will consider is

$$g(x,q) = \begin{cases} \int_x^q (v-x)\,d\lambda(v), & \text{if } x \le q, \\ \int_q^x (x-v)\,d\lambda(v), & \text{if } x > q, \end{cases} \tag{1}$$

where λ is a measure that is mutually absolutely continuous with Lebesgue measure and is finite on every bounded interval. It is helpful to rewrite (1) as

$$g(x,q) = \int_q^x (x-v)\,d\lambda(v), \tag{2}$$

Coherence$_{2+}$: **Allow different (strictly) proper scoring rules $\mathcal{S}_i[X_i, F(X_i)]$ for each random variable X_i with Total Score in each state equal to the sum of the individual scores in that state.**

A (finite) set of forecasts {$F(X)$} are *incoherent*$_{2+}$ if there is a rival *finite set* of forecasts {$F'(X)$} that *uniformly* dominates.

- **Coherence$_{2+}$ has coherence$_2$ as a special case.**

Thus, with *coherence*$_{2+}$ both *uniform dominance* and *elicitation* are addressed with the same device: *forecasting*.

Basic Results Connecting these Two Senses of Coherence

Theorem **(de Finetti, 1974):**

A set of previsions {*P*(*X*)} are *coherent*$_1$

if and only if

the same *forecasts* {*F*(*X*): *F*(*X*) = *P*(*X*)} are coherent$_2$

if and only if

There exists a (finitely additive) probability *P* on Ω such that these quantities are the *P*-Expected values of the corresponding variables

$$\mathcal{E}_P[X] = F(X) = P(X).$$

Theorem **(events: Predd *et al.* 2009) & (SSK 2009); variables (SSK, 2014):**

A set of forecasts {*F*(*X*)} are *coherent*$_{2+}$

if and only if

There exists a (finitely additive) *P* such that $\mathcal{E}_P[X] = F(X)$.

***Question*: How does a *coherent*$_{2+}$ agent forecast in the Dollar-Yen puzzle?**

***Answer* (i):**

With the proper scoring rules and currency specified extraneously, the coherent$_2$ Forecaster behaves exactly as the coherent$_1$ Bookie.

***But Answer* (ii):**

With the proper scoring rules specified extraneously, and where the coherent$_2$ agent may choose the currencies for scoring the Forecaster has a strict preference for matching the currency against the scoring rule so as to have a high probability for a small loss.

Similarly, with the currency specified extraneously but with the scoring rule(s) determined by a choice of the Forecaster, there is strategic choice of the scoring rules.

Or, with the *Forecaster* having both the choice of the scoring rule(s) and matching currencies, there is strategic choice available to *Forecaster*.

- **The Forecaster has strategic choices absent in the *Prevision* game!**

Summary

1. **In the *Prevision* game, the *coherent*$_1$ agent behaves as though she/he has a state-independent utility in the currency of the contracts. It is impossible to identify a privileged currency in the *Prevision* game that reveals the agent's authentic credences/degrees of belief.**

2. **In the *Forecasting* game, when scoring rules and currencies are specified extraneously, the *coherent*$_{2+}$ agent behaves as though she/he has a state-independent utility in the currencies of the individual contracts. It is impossible to identify a privileged currency for forecasting that reveals the agent's authentic degrees of belief.**

3. **In the *Forecasting* game, when the *Forecaster* may choose either/both of the scoring rule(s) or/and the currencies, the *Forecaster* has additional strategic choices that are absent in the *Prevision* game.**
    - **But these strategic choices do *not* reveal the agent's credences!**
    - **The situation is even worse for non-SEU decision makers.**