Optimisation under uncertainty applied to a bridge collision problem

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Abstract

We consider the problem of modelling the load on a bridge pillar when hit by a vehicle. This load depends on a number of uncertain variables, such as the mass of the vehicle and its speed on impact. The objective of our study is to analyse their effect on the load. More specifically, we are interested in finding the minimum distance of the pillar to the side of the road passing under the bridge such that a given constraint on the load is satisfied in 99% of impact cases, i.e., such that the probability of satisfying the constraint is 0.99. In addition, we look for solutions to the following optimisation problem: find the distance that minimises a given cost function while still satisfying a given constraint on the load. This optimisation problem under uncertain constraints is not a well-posed problem, so we turn it into a decision problem under uncertainty. For both problems, we consider two typical cases. In the first, so-called *precise-probability case*, all uncertain variables involved are modelled using probability distributions, and in the second, so-called *imprecise-probability case*, the uncertainty for at least some of the variables (*in casu* the mass) is modelled by an interval of possible values, which is a special imprecise-probabilistic model. In the first case, we compute the joint distribution using simple Monte Carlo simulation, and in the second case, we combine Monte Carlo simulation with newly developed techniques in the field of imprecise probabilities. For the optimisation problem with uncertain constraints, this leads to two distinct approaches with different optimality criteria, namely *maximality* and *maximinity*, which we discuss and compare.

1 Introduction

When designing a bridge one should take into account accidental forces. The pillar can for instance be hit by a vehicle, resulting in damage to the bridge and/or loss of human life. Studying the effect of vehicle impact on the integrity of a bridge is a complex issue, involving dynamic effects, non-linear material behaviour, stiffness of bridge and vehicle, the presence of road restraining systems, shape and roughness of the terrain, mass and speed of the vehicle, impact angle, deceleration, etc.

One problem we describe in this paper consists in finding a smallest distance d, from the side of the road passing under the bridge to its pillars, such that in case of collision, the impact force F_{veh} is smaller than a design force F_{des} in 99% of the cases. In a second problem, a cost function $C(d)$ for building the bridge is minimised. In combining both problems, we are looking for the cost-optimal distance d between the vehicle and bridge pillar that also ensures structural integrity, or equivalently, which satisfies the condition on the design force.

The paper is organised as follows. In Section 2 we give a detailed description of the problem, outlining four different interesting cases, depending on the type of vehicle colliding with the bridge pillar. Section 3 provides a short overview of uncertainty models and optimisation criteria under uncertainty. Since interval uncertainties appear in the parameters of the vehicle in two of the four cases considered, we need to treat it separately and use techniques from imprecise-probability theory in order to give the solution. We subsequently split our main developments into two sections: in Section 4 we solve the collision problem in the precise-probability case where all uncertainty models are probabilistic, and in Section 5 we address the imprecise-probability case where some parameters have interval uncertainty. We present some conclusions in Section 6.

2 Problem description

2.1 Constrained optimisation under uncertainty

We use the approach from earlier work [3], where some of us solve the following general constrained optimisation problem with uncertainty on the constraint:

$$\text{maximise } f(x) \text{ over all } x \in \mathscr{X} \text{ such that } xRY,$$

where f is a bounded real-valued function defined on a set $\mathscr{X}$, Y is a random variable taking values in a set $\mathscr{Y}$, and R is a relation on $\mathscr{X} \times \mathscr{Y}$. First, we model the bridge collision problem as such a constrained optimisation problem under uncertainty. Next, by considering closely all types of uncertainties and conditions, we apply the methodology developed in our earlier work [3] to find the solutions to this problem, which are presented in subsequent sections.

2.2 Vehicle–pillar collision problem

We consider a situation in which a vehicle collides with the pillar of a bridge over the road that the vehicle was travelling on. See Figure 1 for a schematic drawing of the accident situation and the type of bridge we consider in this paper.
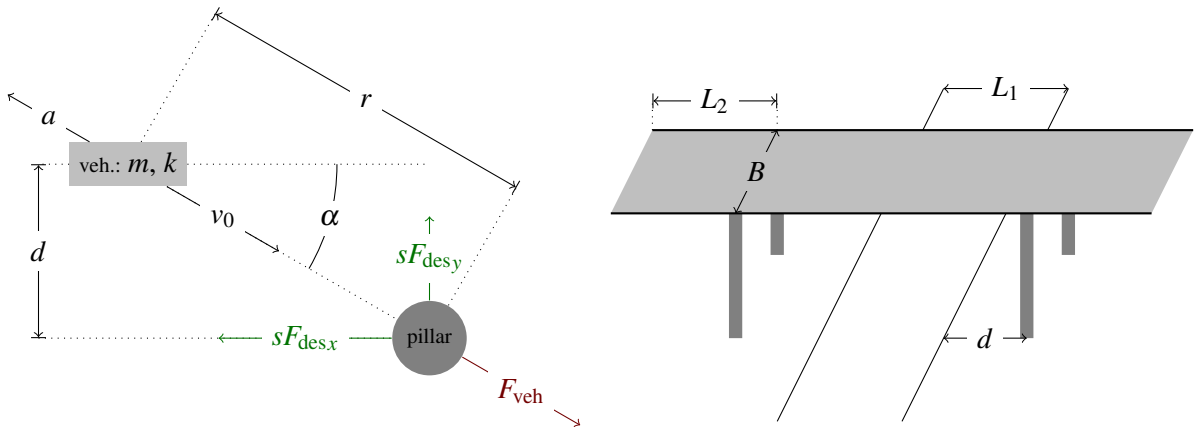

Figure 1: Schematic drawing of the vehicle–pillar collision and the bridge type considered

The first problem we wish to solve is the following [6]: find the smallest distance d from the side of the carriageway to the bridge pillar such that whenever a collision occurs, the probability that the impact force F_{veh} stays below a given safety threshold still exceeds 99%:

$$P\left(F_{\text{veh}}\cos\alpha \leq sF_{\text{des}x} \text{ and } F_{\text{veh}}\sin\alpha \leq sF_{\text{des}y}\right) \geq 0.99. \tag{1}$$

Apart from d, the impact force depends on the mass m of the vehicle, its stiffness k, the speed v_0 at which it leaves the carriageway, its average deceleration a while off the carriageway, and—via the distance r traveled between leaving the carriageway and hitting the pillar—on the angle α at which the vehicle leaves the carriageway:

$$F_{\text{veh}} = \sqrt{mk\left(v_0^2 - 2ar\right)} = \sqrt{mk\left(v_0^2 - 2ad/\sin\alpha\right)}. \tag{2}$$

A safety threshold is specified separately for, respectively, the longitudinal and perpendicular components of the impact force. Each is the product of a force that the pillar is designed to withstand in that direction, $F_{\text{des}x}$ and $F_{\text{des}y}$ respectively, and a factor s that expresses the relation to the corresponding dynamic impact forces.

As a second problem, we consider minimising the cost, expressed in euros, for building a single-span bridge:

$$C(d) = 45B\left((L_1 + 2d)^2 + L_2^2\right). \tag{3}$$

In this expression, B is the width of the bridge, L_1 is the carriageway width, so $L_1 + 2d$ is the main span, and L_2 is the span next to the abutments. Therefore, modelling the bridge collision problem as a constrained optimisation problem leads to:

$$\text{maximise } f(d) \text{ over all } d \in \mathbb{R}^+ \text{ subject to } dRY, \tag{4}$$

where $Y := (m, v_0, a, \alpha)$ is a quadruple of uncertain variables, $\mathbb{R}^+$ is the set of all non-negative real numbers, the relation R is defined by

$$dRY \Leftrightarrow F_{\text{veh}}\cos\alpha \leq sF_{\text{des}x} \text{ and } F_{\text{veh}}\sin\alpha \leq sF_{\text{des}y} \tag{5}$$

and the objective function is

$$f(d) = -45B\left((L_1 + 2d)^2 + L_2^2\right). \tag{6}$$

In what follows, we consider four categories, based on the type of vehicle and the speed when leaving the carriageway. The information we have [6] about the parameters appearing in this problem can be a probability distribution, or given by an interval of reasonable values, or by such an interval and additionally a probability distribution. We use the normal distribution $\mathscr{N}(\mu, \sigma)$, the lognormal distribution $\mathscr{L}(\mu, \sigma)$, and the Rayleigh distribution $\mathscr{R}(\mu, \sigma)$, each of which is characterised by a mean μ and standard deviation σ and must be used truncated to within the given reasonable interval.

The available information about some parameters is constant over the vehicle–speed categories: $k = 300\,\text{kN/m}$, $a \in [1,5]\text{m/s}^2$ and distributed according to $\mathscr{L}(4, 1.3)$, $s = 1.4$, $B = 14\,\text{m}$, $L_1 = 20\,\text{m}$, and $L_2 = 15\,\text{m}$. The different vehicle–speed categories and the parameters along with the information varying over them is given in Table 1. It is assumed that within each category, the component random variables v_0, m, a and α are *independent*.

Parameter	Highway lorry		Urban lorry		Courtyard car		Parking car	
v_0 [km/h]	$[50, 100]$	$\mathscr{L}(80, 10)$	$[30, 70]$	$\mathscr{L}(40, 8)$	$[5, 30]$	$\mathscr{L}(15, 5)$	$[5, 20]$	$\mathscr{L}(5, 5)$
m [t]	$[12, 40]$	$\mathscr{N}(20, 12)$	$[12, 40]$	$\mathscr{N}(20, 12)$	$[0.5, 1.6]$		$[0.5, 1.6]$	
α [°]	$[8, 45]$	$\mathscr{R}(19, 10)$	$[8, 45]$	$\mathscr{R}(10, 10)$	$[8, 45]$	$\mathscr{R}(10, 10)$	$[8, 45]$	$\mathscr{R}(10, 10)$
$F_{\text{des}x}$ [kN]	1000		500		50		40	
$F_{\text{des}y}$ [kN]	500		250		25		25	

Table 1: Parameter value information for the different vehicle–speed categories

3 Uncertainty models and optimisation criteria

Let us now briefly describe how we can transform the optimisation problem with uncertain constraints, described by Eqs. (4)–(6), into a problem of decision making that can be addressed using techniques from the theory of imprecise probabilities. We follow the steps we have already proposed in previous work [3].

Since we are uncertain about the value that the random variable (quadruple) Y assumes, the optimisation problem considered here is not well-posed, and we propose to turn it into a well-posed problem through the following trick: we consider a utility (or gain) function $G_d \colon \mathscr{Y} \to \mathbb{R}$ defined by

$$G_d(y) = L + (f(d) - L)I_{dR} = \begin{cases} f(d), & \text{if } dRy, \\ L, & \text{otherwise,} \end{cases} \tag{7}$$

where L is a constant such that $L < \inf f$, and I_{dR} is the indicator function of the set $dR = \{y \in \mathscr{Y} : dRy\}$—the so-called R-afterset of d, assuming the value one on this set, and zero elsewhere. It associates with every possible *decision*, or *choice* of d, a gain which is equal to f when the constraint is satisfied, and equal to a *constant penalty L* when the constraint is not satisfied.

We are now faced with a decision problem: decide which d to choose in the face of our uncertainty about Y.

When the uncertainty about Y is modelled probabilistically, the approach usually followed is to choose that action d which maximises the expected gain $E(G_d) = L + (f(d) - L)E(I_{dR})$.

As we see, in order to compute $E(G_d)$, we need an evaluation of $E(I_{dR})$. We obtain that by performing a Monte Carlo simulation, which is a widely used computational method for generating probability distributions of variables that depend on other variables or parameters represented as probability distributions. In its most straightforward form, a Monte Carlo simulation usually assumes that input parameters are independent. If they are not, some adjustment may be necessary. As with any use of a mathematical model, the results are only as good as the assumptions, and the choice of assumptions, particularly simplifying ones, requires professional judgement. In our problem, because all the variables are taken to be independent, we can perform a simple Monte Carlo analysis to obtain $E(I_{dR})$.

When more general, so-called imprecise probability models [2, 5] are given for the uncertainty about Y, the linear expectation operator E is replaced by a pair of non-linear lower and upper expectation operators $\underline{E}$ and $\overline{E}$, which provide lower and upper bounds for the expectation E. This will for instance be the case for the courtyard and parking car categories described above, where we have no probabilistic but only interval information for the mass.

In this case, we must extend the ideas behind maximising expected gain towards our more general context. This can be done in two interesting ways. In the first one, called the *maximinity* approach, we choose the decision d which maximises the lower expected gain $\underline{E}(G_d)$. It can be interpreted as follows: we look at the worst that could happen under each decision and then choose the decision with the largest payoff. This criterion leads to *the maximum of the minima* or the *best of the worst*. Taking into account that

$$\underline{E}(G_d) = L + (f(d) - L)\underline{E}(I_{dR}),$$

the solution for this problem is given by:

$$\operatorname*{argsup}_{d \in \mathbb{R}^+} \underline{E}(G_d) = \operatorname*{argsup}_{d \in \mathbb{R}^+}[f(d) - L]\underline{E}(I_{dR}). \tag{8}$$

This means that we need to calculate $\underline{E}(I_{dR})$. If g is any real-valued function of m, v_0, a and α, then $\underline{E}(g)$ is calculated, due to the independence assumptions (see also [5, Chapter 9] and [1]) by first fixing a value of the mass m in the given interval $M = [\underline{m}, \overline{m}]$ of reasonable values, and calculating $E(g(m, \cdot, \cdot, \cdot))$ by Monte Carlo simulation as indicated above, and then minimising over all possible m in the interval M:

$$\underline{E}(g) = \inf_{m \in M} E(g(m, \cdot, \cdot, \cdot)). \tag{9}$$

$\overline{E}(g)$ is found similarly, by maximising over all m in M.

With $\underline{E}(I_{dR})$ computed as in Eq. (9), the problem to solve is now a classical optimisation problem.

For the second, so-called *maximality* approach, we define a strict partial order on the possible decisions as follows:

$$d_1 \succ d_2 \Leftrightarrow \underline{E}(G_{d_1} - G_{d_2}) > 0.$$

This is a *robustified* version of the order on the decisions d considered for precise probability models, which is based on expected gain: d_1 is to be preferred to d_2 if and only if it would be so preferred for all the precise probability models that are compatible with the imprecise model, meaning that their expectation operators E lie between the lower and upper expectation operators $\underline{E}$ and $\overline{E}$.

An optimal decision will now be any $d \in \mathbb{R}^+$ that is undominated in this strict partial order, and it can be shown [3, 5] that this is the case if and only if $\overline{E}(G_d - G_e) \geq 0$ for all $e \in \mathbb{R}^+$, or equivalently,

$$\inf_{e \in \mathbb{R}^+} \overline{E}\left([f(d) - L]I_{dR} - [f(e) - L]I_{eR}\right) \geq 0. \tag{10}$$

If we use the methodology leading to Eq. (9), we see that we need to consider the function

$$H(d, e) = \sup_{m \in M} \left[[f(d) - L]E(I_{dR}(m, \cdot, \cdot, \cdot)) - [f(e) - L]E(I_{eR}(m, \cdot, \cdot, \cdot))\right], \tag{11}$$

where, as before, the expectations can be calculated using Monte Carlo methods. An optimal distance d is such that $H(d, e) \geq 0$ for all $e \in \mathbb{R}^+$. We can find such d by looking at plots of the function $H(d, e)$.

Observe that this second method will generally lead to an interval of optimal solutions, each member of which is *incomparable* (nor better, nor worse, nor equivalent) with the others: given the available information about the random variable Y, these are the optimal values, but nothing more can be said to allow us to refine this picture.

It can be shown [5] that the maximinity solution is always included in the maximality interval of solutions, and that in the case of precise probability models for Y, both solution methods coincide with maximising expected gain.

This provides us with two types of solutions to the cost optimisation problem. Of course we also need to check whether these optimal solutions satisfy the safety constraint (1), and therefore dominate the smallest value of the distance that does so.

When the available model for Y is (precise-)probabilistic, the probability on the left-hand side in Eq. (1) is given by $E(I_{dR})$, and can—as explained above—be found by simple Monte Carlo methods.

For the courtyard and parking car categories, we only have interval information for the mass, and then we must impose that all compatible precise models (one for each possible value of the mass m) should satisfy the constraint, which is equivalent to requiring that $\underline{E}(I_{dR}) \geq 0.99$, where as before, the left-hand side can be calculated by Monte Carlo methods and subsequent minimisation over all m in M.
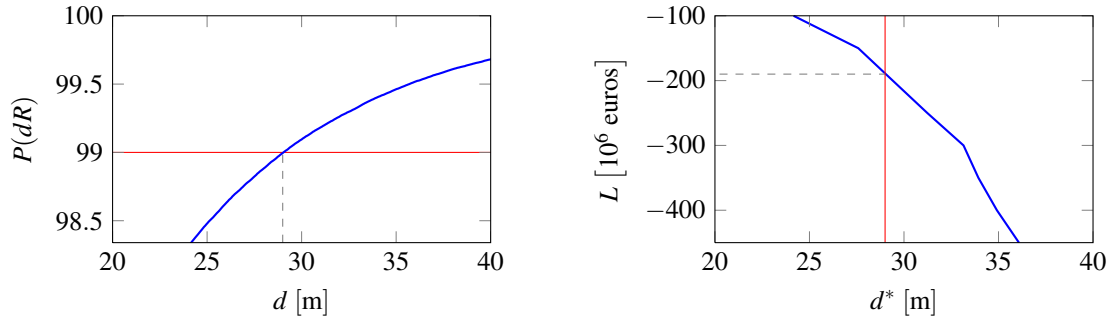
4 Vehicle impact on bridge pillar: Precise-probabilistic case

Let us now succinctly present the numerical results of applying the methods outlined in the previous section to the four categories given in Section 2. We start with the case where all uncertainty models for the random variables appearing in Eq. (1)—mass m, initial speed v_0, deceleration a and angle α—are (precise) probability models. The information we have about these models can be found in Table 1 and the paragraph preceding it.

4.1 Highway lorry

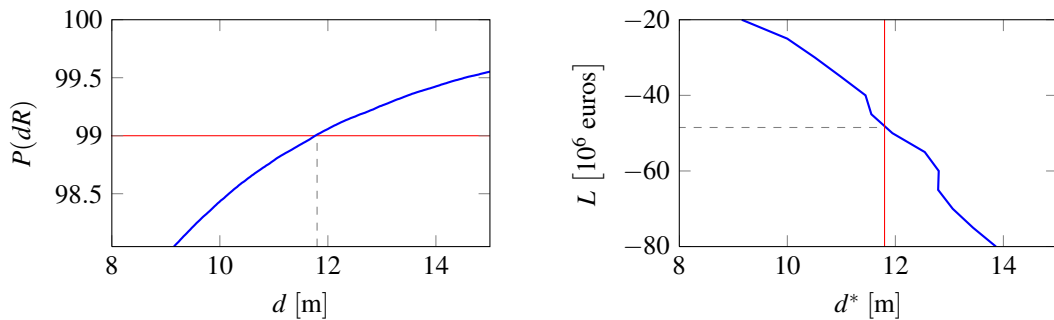First, we look at the fastest and heaviest type of vehicle: the highway lorry.

The graph of the probability $P(dR) = E(I_{dR})$ that the constraint (1) will be satisfied as a function of d is given on the left-hand side below. It allows us to pinpoint $d_{\min} \approx 29\,\text{m}$ as the smallest acceptable value for d.

In addition, we give the graph of the optimal value d^* as a function of the penalty L (calculated with a step size of 50 and expressed in millions of euros), but with inverted axes to allow for easy comparison with the previous graph. It appears that for penalties L greater in magnitude than about 200 million euros, the optimal value d^* satisfies the constraint (1). This gives us an idea of the penalty implied by the somewhat arbitrary 99% borderline.

4.2 Urban lorry

For the urban lorry category, we see that $d_{\min} \approx 12\,\text{m}$. There is also the following relationship between the optimal value d^* and the penalty L (calculated with a step size of 5 and expressed in millions of euros). For penalties L greater in magnitude than about 50 million euros, the optimal value d^* of d satisfies the constraint (1).
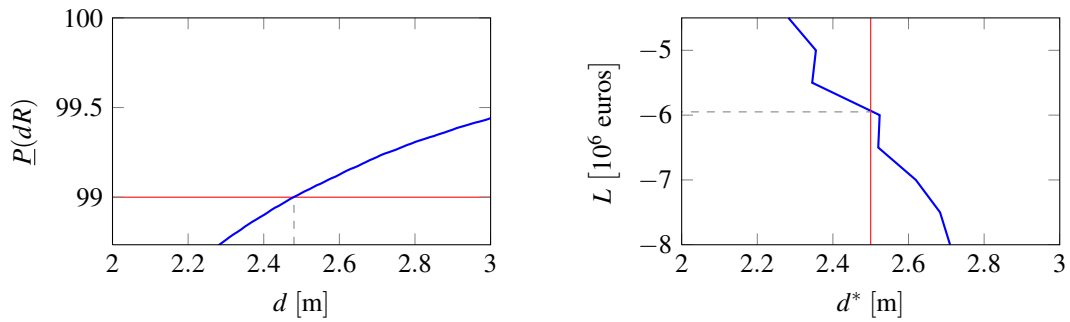

5 Vehicle impact on bridge pillar: Imprecise-probabilistic case
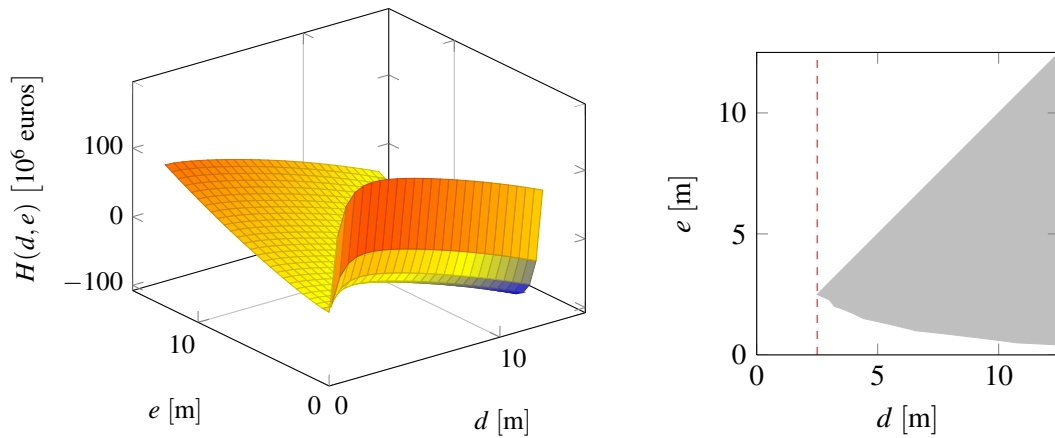
In this section, we look at the categories for which our information about one of the parameters, namely the mass m, is given by an interval. To deal with it, we use the techniques from the theory of imprecise probabilities elaborated on in Section 3. We explained there that two possible approaches can be taken to decide which d-values are optimal or not, maximinity and maximality; both will be used. Again, the information we have about the parameter values can be found in Table 1 and the paragraph above it.

5.1 Courtyard car

Maximinity   The following plots represent the lower probability $\underline{P}(dR) = \underline{E}(I_{dR})$ of satisfying the force constraint as a function of d (left-hand side), and the relation between the cost-optimal distance d^* and the penalty L (calculated with a step size of 0.5 and expressed in millions of euros). We see that $d_{\min} \approx 2.5\,\mathrm{m}$, and that for penalties greater in magnitude than about 6 million euros, the cost-optimal distance d^* under maximinity guarantees that the constraint (1) is satisfied.
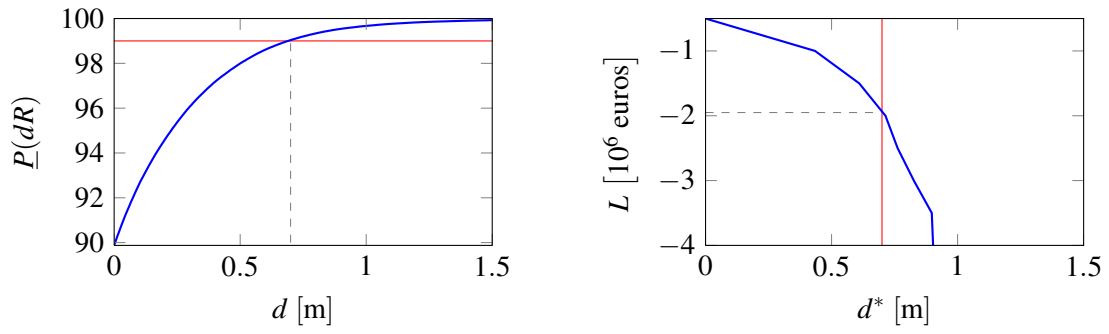

Maximality   For the solutions involving the maximality criterion, we need to consider the function $H(d,e)$; we have depicted this function, for $L = 6$ million euros, in the plot below on the left-hand side. In the contour plot on the right, we have depicted in grey the region where H is negative. This leads to the interval $[0\,\mathrm{m}, 2.5\,\mathrm{m}]$ of optimal values for d, which as expected, contains the maximin cost-optimal solution.
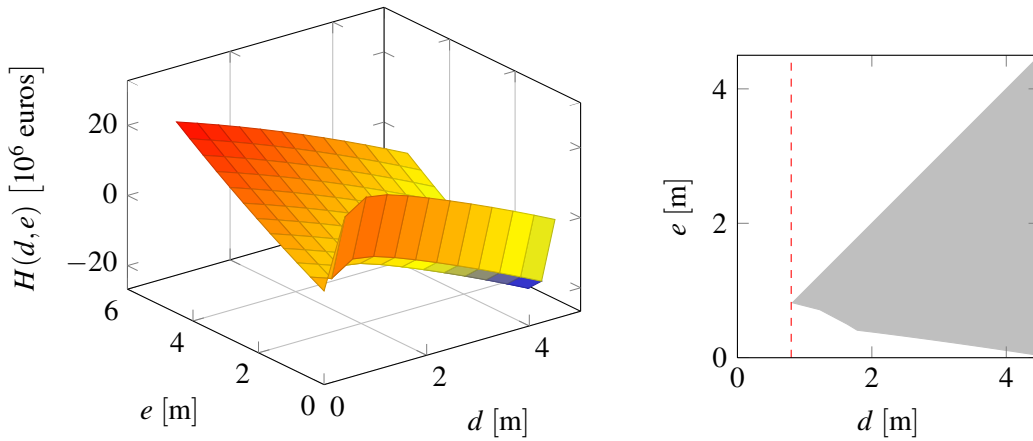

5.2 Parking car

Maximinity   The following plots represent the lower probability $\underline{P}(dR) = \underline{E}(I_{dR})$ of satisfying the force constraint as a function of d (left-hand side), and the relation between the cost-optimal distance d^* and the penalty L (calculated with a step size of 0.5 and expressed in millions of euros). We can see that $d_{\min} \approx 0.7\,\mathrm{m}$, and that for penalties greater in magnitude than about 2 million euros, the cost-optimal distance d^* under maximinity guarantees that the constraint (1) is satisfied.

Maximality For the solutions involving the maximality criterion, we again need to consider the function $H(d,e)$; we have depicted this function, for $L = 2$ million euros, in the plot below on the left-hand side. In the contour plot on the right, we have depicted in grey the region where H is negative. This leads to the interval $[0\,\mathrm{m}, 0.7\,\mathrm{m}]$ of optimal values for d, which as expected, contains the maximin cost-optimal solution.

6 Conclusions

We have considered an optimisation problem under uncertain constraints, where (i) the optimal distance from the side of the road where a bridge pillar should be built has to be found such that the probability of the impact force in case of collision being smaller than a design value is greater than 99%. At the same time, (ii) we addressed the minimisation of the cost function of building the bridge. The usual optimisation techniques could not be applied directly because of the presence of uncertain constraints in the model. A new approach was taken in order to remedy this problem, which converts the optimisation into a decision making problem under uncertainty.

We considered four categories of vehicles, and in each case we have found a cost-optimal distance from the side of the road where a bridge should be built such that the given constraints are satisfied.

The distance d from pillar to road can also be chosen in accordance with what the area between road and pillar is used for. This leads to the following course of reasoning: the *functionality distance* d_{fun} is the sum of the widths d_1 of the emergency lane, the width d_2 of the dewatering facilities, and the width d_3 of the maintenance road or inspection path. For a motorway, usually $d_1 = 3\,\mathrm{m}$, $d_2 = 1$ to $2\,\mathrm{m}$ and $d_3 = 1$ to $5\,\mathrm{m}$. This yields a functionality distance d_{fun} between 5 and $10\,\mathrm{m}$.

In the case of the highway lorry, the cost-optimal distance of $29\,\mathrm{m}$ exceeds the maximal functionality distance of $10\,\mathrm{m}$. A possible explanation for the difference can to an important degree be attributed to our letting $s = 1.4$ in the calculations. The factor s converts the highest allowable elastic design force into the highest allowable dynamic design force. Therefore this s depends quite strongly on the actual design of the pillars.

A commonly used value for s is 2.4. At a value of 2.4 the corresponding cost-optimal distance d^* would be smaller than the one for $s = 1.4$, and it seems clear that our optimisation tool can be used to determine the value of s for which $d^* \approx d_{\text{fun}}$. We could for instance then design the pillar in such a way that the dynamic resistance would be at least s times the elastic resistance.

For the urban lorry case, the cost-optimal distance is 12 m, which is of the same order as the functionality distance.

In practice the available space next to the road is limited, and the minimal functionality distance 5 m is used. In both cases (highway lorry and urban lorry), the cost-optimal distance is much higher than the minimal functionality distance. When using the minimal functionality distance, road restraint systems are used in front of the pillar because the actual collision force is now much more likely to exceed the design force.

For the parking car category, the cost-optimal distance of 0.7 m corresponds to the safe object distance of 1 m that is used in road design.

The cost-optimal distance of 2.5 m for the courtyard car category indicates that the expected accidental force is higher than the design force. Of course, in practice the courtyard car and the parking car categories are not considered in bridge design, due to the much smaller design force (50 kN) in comparison with the highway lorry category (1000 kN). However, when designing structures at the edge of parking areas, one should be aware that the design forces are too low, as our study indicates, and therefore protective measures should be considered.

Acknowledgements

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