A constrained optimization problem under uncertainty

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I. Constrained optimization problem under uncertainty

General case	Simple example
maximize $f(x)$ subject to xRY	maximize x subject to $x \leq Y$
- f is a bounded function $f: \mathscr{X} \to \mathbb{R}$ over all x in $\mathscr{X}$ - Y is a random variable taking values y in a set $\mathscr{Y}$ - R is a relation on $\mathscr{X} \times \mathscr{Y}$	- $x \in (m, +\infty) \subseteq \mathscr{X} := \mathbb{R}$ - Y is a real random variable

Maximize a bounded function $f: \mathscr{X} \to \mathbb{R}$ over all $x \in \mathscr{X}$ that satisfy the constraint xRY

II. Uncertainty Models

Coherent lower prevision

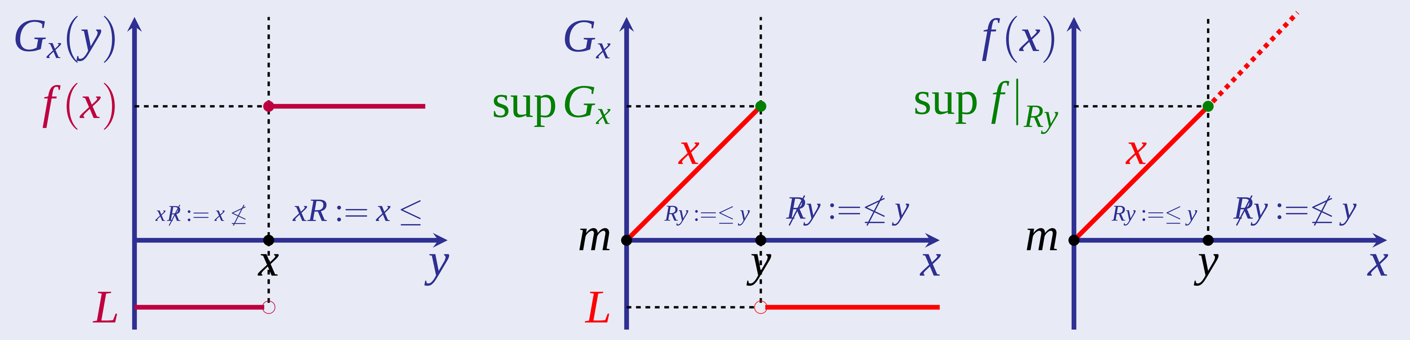
$$\underline{P}_Y(g) = \inf\{P(g) : P \in \mathscr{M}\}, \quad g: \mathscr{Y} \to \mathbb{R} \quad \text{and} \quad \mathscr{M} \text{ is a credal set.}$$

Different types of uncertainty models for the random variable Y:

Model	Credal set
Linear previsions P	$\mathscr{M} = \{P\}$
Vacuous previsions on A	$\underline{P}_Y(g) = \inf_{y \in A \subseteq \mathscr{Y}} g(y)$ and $\mathscr{M} = \{P : P(A) = 1\}$
Possibility distributions π	$\mathscr{M} := \{P : \forall B \subseteq \mathscr{Y},\ P(B) \leq \Pi(B),\ \Pi(B) = \sup_{y \in B} \pi(y)\}$
Probability boxes $\underline{F}, \overline{F}$	$\mathscr{M} := \{P : \underline{F}(y) := \underline{P}(Y \leq y) \leq P(Y \leq y) \leq \overline{F}(y) := \overline{P}(Y \leq y)\}$

III. Utility function

General case	Simple example
$\forall x \in \mathscr{X},\quad G_x : \mathscr{Y} \to \mathbb{R},\quad L < \inf f$	$\forall x \in (m, +\infty),\quad G_x : \mathbb{R} \to \mathbb{R},\quad L < m$
$G_x(y) := (f(x) - L) I_{xR}(y) + L = \begin{cases} f(x) & \text{if } xRy \\ L & \text{if } x\not R y \end{cases}$	$G_x(y) := (x - L) I_{x\leq}(y) + L = \begin{cases} x & \text{if } x \leq y \\ L & \text{if } x \not\leq y \end{cases}$



$Ry := \{x \in \mathscr{X} : xRy\}$
$xR := \{y \in \mathscr{Y} : xRy\}$
$\leq y := (m, y]$
$x \leq := [x, +\infty)$

IV. Decision making

Decision Criteria

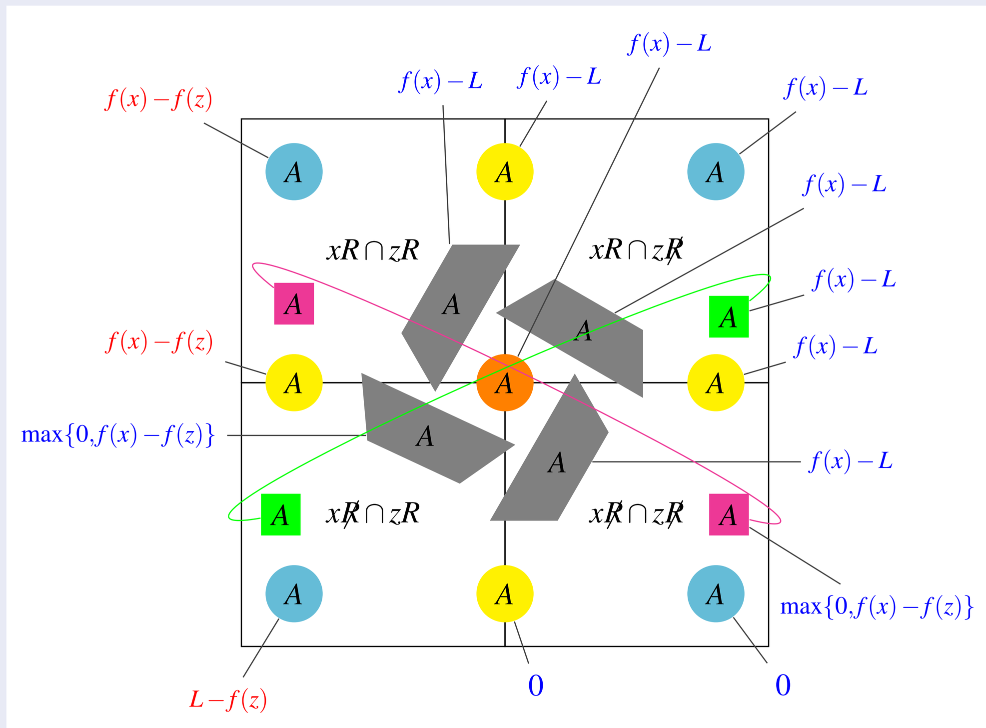
Maximinity (Γ-maximin)	Maximality
$x \succ z \Leftrightarrow \forall z \in \mathscr{X},\ \underline{P}_Y(G_x) > \underline{P}_Y(G_z)$	$x \succ z \Leftrightarrow \forall z \in \mathscr{X},\ \underline{P}_Y(G_x - G_z) > 0$

Reformulation of original problem as decision problem under uncertainty

with the maximinity criterion	with the maximality criterion
Pick the $x \in \mathscr{X}$ that maximize $\underline{P}_Y(G_x)$	Pick the $x \in \mathscr{X}$ for which $\min_{z \in \mathscr{X}} \overline{P}_Y(G_x - G_z) \geq 0$

V. Solutions

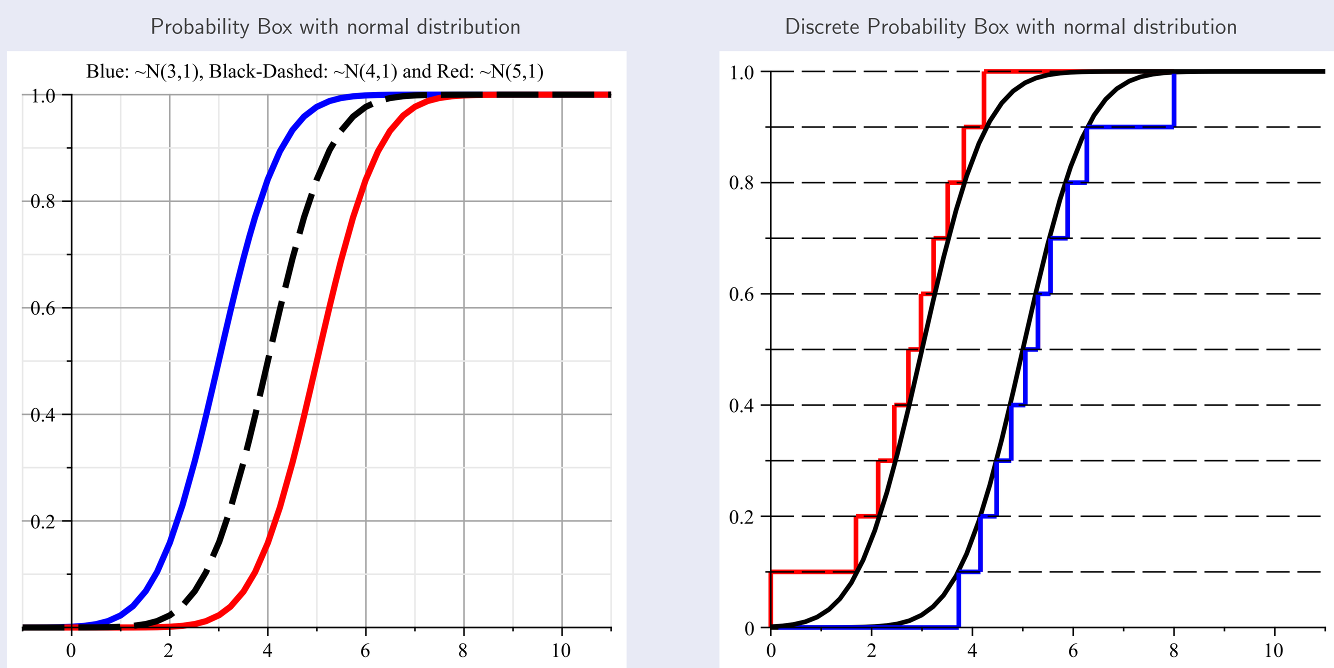
Criterion	Case	Linear Prevision	Vacuous Prevision
Maximinity	General	$\operatorname{argmax}_{x \in \mathscr{X}} [(f(x) - L) P_Y(I_{xR})]$	$(\underline{RA} = \emptyset) \vee \left(x \in \overline{RA} \wedge f(x) \geq \max f\|_{\underline{RA}}\right)$ $\underline{RA} := \bigcap_{y \in A} Ry \quad \overline{RA} := \bigcup_{y \in A} Ry$
	Simple	$x = \left\{\frac{b + \min(\max(m,a),b)}{2}\right\}$	$\operatorname{argmax}_{x \leq a} x = \{a\}$
Maximality	General	$\operatorname{argmax}_{x \in \mathscr{X}} [(f(x) - L) P_Y(I_{xR})]$	$\operatorname{argmax}_{x \in \mathscr{X}} [(f(x) - L) P_Y(I_{xR})]$
	Simple	$\operatorname{argmax}_{x \in (m,+\infty)} [(x - L) P_Y(I_{x\leq})]$	$\left(x \leq b \wedge x \geq \max_{z \leq a} z\right) \Rightarrow x \in [\max(m,a), b]$



$$\overline{P}_Y(G_x - G_z) = \sup_{y \in A} [(f(x) - L) I_{xR}(y) - (f(z) - L) I_{zR}(y)]$$

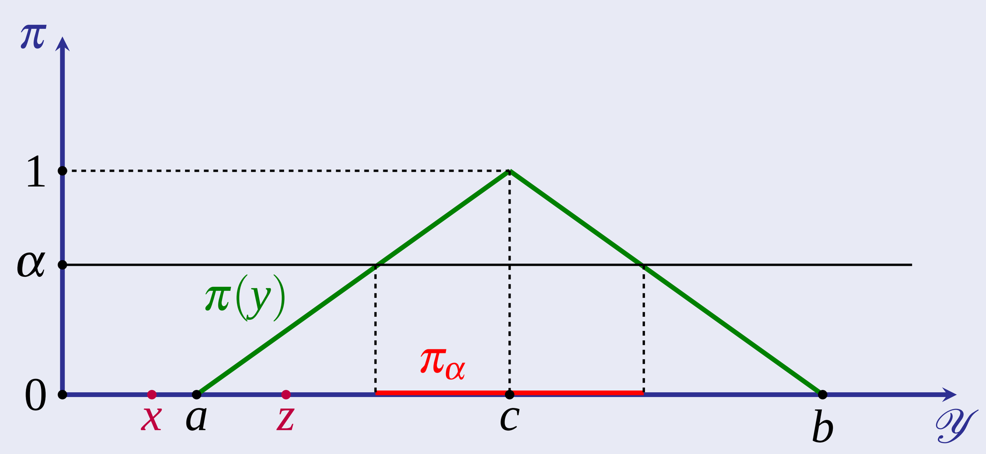
Now we choose $\mathscr{X} = \mathscr{Y} = \mathbb{R}$ and $R = \leq$ for the general case as well

Criterion	Case	Possibility Distribution	P-box with uniform distribution
Maximinity	General	$\operatorname{argmax}_{x \in \mathbb{R}} \begin{cases} f(x) & x \leq a \\ L\frac{x-a}{c-a} + f(x)\frac{c-x}{c-a} & a < x < c \end{cases}$	$\operatorname{argmax}_{x \in \mathbb{R}} \begin{cases} f(x) & x \leq a \\ L\frac{x-a}{d-a} + f(x)\frac{d-x}{d-a} & a < x < d \end{cases}$
	Simple	$\operatorname{argmax}_{x \in (m,+\infty)} \begin{cases} x & x \leq a \\ L\frac{x-a}{c-a} + x\frac{c-x}{c-a} & a < x < c \end{cases}$	$\operatorname{argmax}_{x \in (m,+\infty)} \begin{cases} x & x \leq a \\ L\frac{x-a}{d-a} + x\frac{d-x}{d-a} & a < x < d \end{cases}$
Calculation	Simple	$x = \left\{\frac{c + \min(\max(m,a),c)}{2}\right\}$	$x = \left\{\frac{d + \min(\max(m,a),d)}{2}\right\}$

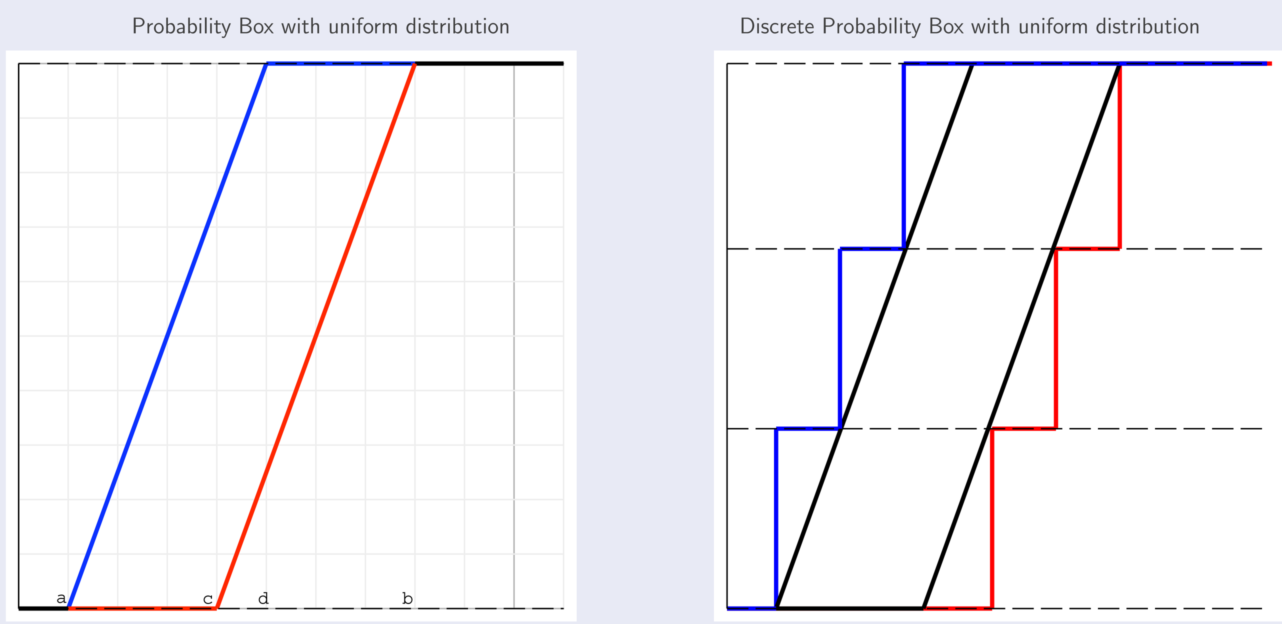


Triangular possibility distribution

$$\underline{P}_Y(G_x) = \begin{cases} f(x) & x \leq a \\ \int_0^{\frac{x-a}{c-a}} L\, d\alpha + \int_{\frac{x-a}{c-a}}^1 f(x) d\alpha & a < x \leq c \\ L & c < x \leq b \vee x > b \end{cases}$$



$$\pi_\alpha = \{y \in \mathscr{Y} : \pi(y) \geq \alpha\}$$



the blue line $\sim U(a,d) := \overline{F}$ (Upper CDF) and the red line $\sim U(c,b) := \underline{F}$ (Lower CDF)

The blue line is a discrete upper bound for $\overline{F}$ and the red line is a discrete lower bound for $\underline{F}$

$$\underline{P}_Y(G_x) = \begin{cases} f(x) & x \leq a \\ L\frac{x-a}{d-a} + f(x)\frac{d-x}{d-a} & a \leq x < d \\ L & \text{else} \end{cases}$$