Exchangeability for sets of desirable gambles

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Abstract

Sets of desirable gambles constitute a quite general type of uncertainty model with an interesting geometrical interpretation. We study exchangeability assessments for such models, and prove a counterpart of de Finetti's finite representation theorem. We show that this representation theorem has a very nice geometrical interpretation. We also lay bare the relationships between the representations of updated exchangeable models, and discuss conservative inference (natural extension) under exchangeability.

Keywords. desirability, real desirability, weak desirability, sets of desirable gambles, coherence, exchangeability, representation, natural extension, updating.

1 Introduction

In this paper, we bring together desirability, an interesting approach to modelling uncertainty, with exchangeability, a structural assessment for uncertainty models that is important for inference purposes.

Desirability, or the theory of (coherent) sets of desirable gambles, has been introduced with all main ideas present—as far as our search has unearthed—by Williams [18, 19, 20]. Building on de Finetti's betting framework [6], he considered the 'acceptability' of *one-sided* bets instead of *two-sided* bets. This relaxation leads one to work with cones of bets instead of with linear subspaces of them. The germ of the theory was, however, already present in Smith's work [15, p. 15], who used a (generally) open cone of 'exchange vectors' when talking about currency exchange. Both authors influenced Walley [16, Sec. 3.7 and App. F], who describes three variants (almost, really, and strictly desirable gambles) and emphasises the conceptual ease with which updated and posterior models can be obtained in this framework [17]. Moral [12, 13] then took the next step and applied the theory to study epistemic irrelevance, a structural assessment. De Cooman and Miranda [1] made a general study of transformational symmetry assessments for desirable gambles.

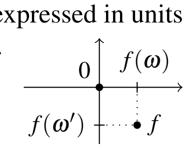
The structural assessment we are interested in here, is exchangeability. Conceptually, it says that the order of the samples in a sequence of them is irrelevant for inference purposes. The first detailed study of this concept was made by de Finetti [4], using the terminology of 'equivalent' events. He proved the now famous Representation Theorem, which is often interpreted as stating that a sequence of random variables is exchangeable if it is conditionally independent and identically distributed. Other important work—all using probabilities or previsions—was done by, amongst many others, Hewitt and Savage [9], Heath and Sudderth [8], and Diaconis and Freedman [7]. Exchangeability in the context of imprecise-probability theory—using lower previsions—was studied by Walley [16, Sec. 9.5] and more in-depth by De Cooman et al. [1–3]. The first embryonic study of exchangeability using desirability was recently performed by Quaeghebeur [14, Sec. 3.1.1].

In this paper, we present the first results of a more matured study of exchangeability using sets of desirable gambles.[1] First, in Sec. 2, we introduce the basics of the theory of desirable gambles. Then, in Sec. 3, we give a desirability-based analysis of finite exchangeable sequences, presenting a Representation Theorem and treating the issues of natural extension and updating under exchangeability.

2 Desirability

Consider a non-empty set Ω describing the possible and mutually exclusive outcomes of some experiment. We also consider a subject, who is uncertain about the outcome of the experiment.

A *gamble* f is a bounded real-valued map on Ω, and it is interpreted as an uncertain reward. When the actual outcome of the experiment is ω, then the corresponding (possibly negative) reward is $f(\omega)$, expressed in units of some pre-determined linear utility. This is illustrated for $\Omega = \{\omega, \omega'\}$. $\mathscr{G}(\Omega)$ denotes the set of all gambles on Ω.



[1] Proofs of this paper's results are included in Appendix A.

We say that a non-zero gamble f is *desirable* to a subject if he accepts to engage in the following transaction, where: (i) the actual outcome ω of the experiment is determined, and (ii) he receives the reward $f(\omega)$, i.e., his capital is changed by $f(\omega)$. The zero gamble is not considered to be desirable.[2]

2.1 Sets of desirable gambles

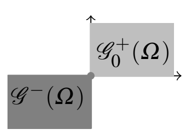
We try and model the subject's beliefs about the outcome of the experiment by considering which gambles are desirable for him. Suppose the subject has a set $\mathscr{R} \subseteq \mathscr{G}(\Omega)$ of desirable gambles.[3]

Definition 1 (Avoiding non-positivity and coherence)**.** *We say that a set of desirable gambles $\mathscr{R}$* avoids non-positivity *if $f \nleq 0$ for all gambles f in* $\mathrm{coni}(\mathscr{R})$.[4] *Let $\mathscr{K}$ be a linear subspace of $\mathscr{G}(\Omega)$ such that $\mathscr{R} \subseteq \mathscr{K}$. Then we say that $\mathscr{R}$ is* coherent relative to $\mathscr{K}$ *if it satisfies the following* rationality requirements*, for all gambles f_1 and f_2 in $\mathscr{K}$ and all real $\lambda > 0$:*

D1. *if $f = 0$ then $f \notin \mathscr{R}$;*
D2. *if $f > 0$ then $f \in \mathscr{R}$ [accepting partial gain];*
D3. *if $f \in \mathscr{R}$ then $\lambda f \in \mathscr{R}$ [scaling];*
D4. *if $f_1, f_2 \in \mathscr{R}$ then $f_1 + f_2 \in \mathscr{R}$ [combination].*

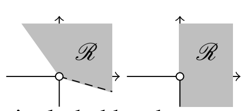
If $\mathscr{R}$ is coherent relative to $\mathscr{G}(\Omega)$, then we simply say that $\mathscr{R}$ is coherent*. We denote the set of coherent sets of desirable gambles by* $\mathbb{D}(\Omega)$.

Requirements D3 and D4 make $\mathscr{R}$ a *cone*: $\mathrm{coni}(\mathscr{R}) = \mathscr{R}$. Due to D2, it includes the positive gambles $\mathscr{G}_0^+(\Omega)$; due to D1, D2 and D4, it excludes the non-positive gambles $\mathscr{G}^-(\Omega)$:



D5. if $f \leq 0$ then $f \notin \mathscr{R}$.

We give two illustrations, the first is a general one and the second models certainty about ω happening. The dashed line indicates a non-included border.



[2]The nomenclature in the literature regarding desirability is somewhat confusing, and we have tried to resolve some of the ambiguity here. Our notion of desirability coincides with Walley's later [17] notion of desirability, initially also used by Moral [12]. Walley in his book [16, App. F] and Moral in a later paper [12] use another notion of desirability. The difference between the two approaches resides in whether the zero gamble is assumed to be desirable or not. We prefer to use the non-zero version here, because it is better behaved in conjunction with our notion of weak desirability in Definition 2.

[3]We use this convention throughout: subscripting a set with zero corresponds to removing zero (or the zero gamble) from the set, if present. For example $\mathbb{R}^+$ ($\mathbb{R}_0^+$) is the set of non-negative (positive) real numbers including (excluding) zero. Further notational conventions: $f \geq g$ iff $f(\omega) \geq g(\omega)$ for all ω in Ω; $f > g$ iff $f \geq g$ and $f \neq g$. The *conical hull operator* coni generates the set of (strictly!) positive linear combinations of elements of its argument set.

[4]A related, but weaker condition, is that $\mathscr{R}$ *avoids partial loss*, meaning that $f \nless 0$ for all gambles f in $\mathrm{coni}(\mathscr{R})$. We need the stronger condition because we have excluded the zero gamble from being desirable.

The intersection $\bigcap_{i \in I} \mathscr{R}_i$ of an arbitrary non-empty family of sets of desirable gambles $\mathscr{R}_i$, $i \in I$, is still coherent. This is the idea behind the following result.

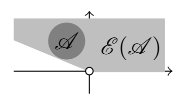
Theorem 1 (Natural extension)**.** *Consider an assessment, a set $\mathscr{A}$ of gambles on Ω, and define its* natural extension

$$\mathscr{E}(\mathscr{A}) := \bigcap \{\mathscr{R} \in \mathbb{D}(\Omega) \colon \mathscr{A} \subseteq \mathscr{R}\} \tag{1}$$
$$= \mathrm{coni}\big(\mathscr{G}_0^+(\Omega) \cup \mathscr{A}\big) \tag{2}$$

Then the following statements are equivalent:

(i) *$\mathscr{A}$ avoids non-positivity;*
(ii) *$\mathscr{A}$ is included in some coherent set of desirable gambles;*
(iii) *$\mathscr{E}(\mathscr{A}) \neq \mathscr{G}(\Omega)$;*
(iv) *$\mathscr{E}(\mathscr{A})$ is a coherent set of desirable gambles;*
(v) *$\mathscr{E}(\mathscr{A})$ is the smallest coherent set of desirable gambles that includes $\mathscr{A}$.*

With a small illustration, we can visualise natural extension as a conical hull operation:



2.2 Weakly desirable gambles, previsions & marginally desirable gambles

We now define *weak desirability*: a useful modification of Walley's [16, Section 3.7] notion of *almost-desirability*. Our conditions for a gamble f to be weakly desirable are more stringent than Walley's for almost-desirability: he only requires that adding any constant strictly positive amount of utility to f should make the resulting gamble desirable. We require that adding anything desirable (be it constant or not) to f should make the resulting gamble desirable. Weak desirability is better behaved under updating: we shall see in Proposition 12 that it makes sure that the exchangeability of a set of desirable gambles, whose definition hinges on the notion of weak desirability, is preserved under updating after observing a sample. This is not necessarily true if weak desirability is replaced by almost-desirability in the definition of exchangeability, as was for instance done in our earlier work [1].

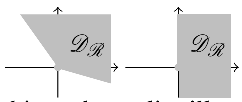
Definition 2 (Weak desirability)**.** *Consider a coherent set $\mathscr{R}$ of desirable gambles. Then a gamble f is called* weakly desirable *if $f + f'$ is desirable for all desirable f', i.e., if $f + f' \in \mathscr{R}$ for all f' in $\mathscr{R}$. We denote the set of weakly desirable gambles by $\mathscr{D}_{\mathscr{R}}$:*

$$\mathscr{D}_{\mathscr{R}} = \{f \in \mathscr{G}(\Omega) \colon f + \mathscr{R} \subseteq \mathscr{R}\}. \tag{3}$$

In particular, every desirable gamble is also weakly desirable, so $\mathscr{R} \subseteq \mathscr{D}_{\mathscr{R}}$.

Proposition 2. *Let $\mathscr{R}$ be a coherent set of desirable gambles, and let $\mathscr{D}_{\mathscr{R}}$ be the associated set of weakly desirable gambles. Then $\mathscr{D}_{\mathscr{R}}$ has the following properties, for all gambles f_1 and f_2 in $\mathscr{G}(\Omega)$ and all real $\lambda \geq 0$:*

WD1. *if* $f < 0$ *then* $f \notin \mathscr{D}_{\mathscr{R}}$ *[avoiding partial loss];*[5]
WD2. *if* $f \geq 0$ *then* $f \in \mathscr{D}_{\mathscr{R}}$ *[accepting partial gain];*
WD3. *if* $f \in \mathscr{D}_{\mathscr{R}}$ *then* $\lambda f \in \mathscr{D}_{\mathscr{R}}$ *[scaling];*
WD4. *if* $f_1, f_2 \in \mathscr{D}_{\mathscr{R}}$ *then* $f_1 + f_2 \in \mathscr{D}_{\mathscr{R}}$ *[combination].*



Like $\mathscr{R}$, $\mathscr{D}_{\mathscr{R}}$ is a cone, but it always includes all cone surface gambles (excluding those that incur a partial loss). We have applied this to the earlier illustrations; take note of border changes.

With a set of gambles $\mathscr{A}$, we associate a *lower prevision* $\underline{P}_{\mathscr{A}}$ and an *upper prevision* $\overline{P}_{\mathscr{A}}$ by letting

$$\underline{P}_{\mathscr{A}}(f) = \sup\{\mu \in \mathbb{R} \colon f - \mu \in \mathscr{A}\} \tag{4}$$

$$\overline{P}_{\mathscr{A}}(f) = \inf\{\mu \in \mathbb{R} \colon \mu - f \in \mathscr{A}\} \tag{5}$$

for all gambles f. Observe that $\underline{P}_{\mathscr{A}}$ and $\overline{P}_{\mathscr{A}}$ always satisfy the *conjugacy relation* $\underline{P}_{\mathscr{A}}(-f) = -\overline{P}_{\mathscr{A}}(f)$. We call a real functional $\underline{P}$ on $\mathscr{G}(\Omega)$ a *coherent lower prevision* if and only if there is some coherent set of desirable gambles $\mathscr{R}$ on $\mathscr{G}(\Omega)$ such that $\underline{P} = \underline{P}_{\mathscr{R}}$.

Theorem 3. *Let $\mathscr{R}$ be a coherent set of desirable gambles. Then $\underline{P}_{\mathscr{R}}$ is real-valued, $\underline{P}_{\mathscr{R}} = \underline{P}_{\mathscr{D}_{\mathscr{R}}}$, $\underline{P}_{\mathscr{R}}(f) \geq 0$ for all $f \in \mathscr{D}_{\mathscr{R}}$. Moreover, a real functional $\underline{P}$ is a coherent lower prevision iff it satisfies the following properties, for all gambles f_1 and f_2 in $\mathscr{G}(\Omega)$ and all real $\lambda \geq 0$:*

P1. $\underline{P}(f) \geq \inf f$ *[accepting sure gain];*
P2. $\underline{P}(f_1 + f_2) \geq \underline{P}(f_1) + \underline{P}(f_2)$ *[super-additivity];*
P3. $\underline{P}(\lambda f) = \lambda \underline{P}(f)$ *[non-negative homogeneity].*

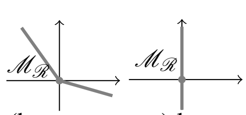
Finally, we turn to marginal desirability. Given a coherent set of desirable gambles $\mathscr{R}$, we define the associated set of *marginally desirable* gambles as

$$\mathscr{M}_{\mathscr{R}} := \{f - \underline{P}_{\mathscr{R}}(f) \colon f \in \mathscr{G}(\Omega)\}\,. \tag{6}$$

The set of marginally desirable gambles $\mathscr{M}_{\mathscr{R}}$ is completely determined by the lower prevision $\underline{P}_{\mathscr{R}}$. The converse is also true:

Proposition 4. *Let $\mathscr{R}$ be a coherent set of desirable gambles. Then $\underline{P}_{\mathscr{M}_{\mathscr{R}}} = \underline{P}_{\mathscr{R}}$ and*

$$\mathscr{M}_{\mathscr{R}} = \mathscr{M}_{\underline{P}_{\mathscr{R}}} := \{f \in \mathscr{G}(\Omega) \colon \underline{P}_{\mathscr{R}}(f) = 0\}\,. \tag{7}$$



The set of marginally desirable gambles $\mathscr{M}_{\mathscr{R}}$ is the entire cone surface of $\mathscr{R}$ and $\mathscr{D}_{\mathscr{R}}$, possibly including gambles that incur a partial (but not a sure) loss.

2.3 Updating sets of desirable gambles

Consider a set of desirable gambles $\mathscr{R}$ on Ω. With a non-empty subset B of Ω, we associate an *updated* set of desirable gambles on Ω, as defined by Walley [17]:

$$\mathscr{R} \| B := \{f \in \mathscr{G}(\Omega) \colon I_B f \in \mathscr{R}\}\,. \tag{8}$$

[5]Compare this to the less stringent requirement for almost-desirability [16, Section 3.7.3]: if $f \in \mathscr{D}_{\mathscr{R}}$ then $\sup f \geq 0$ [avoiding sure loss].

We find it more convenient to work with the following, slightly different but completely equivalent, version:

$$\mathscr{R} | B := \{f \in \mathscr{R} \colon I_B f = f\} = \mathscr{R} \cap \mathscr{G}(\Omega)|B, \tag{9}$$

which completely determines $\mathscr{R}\|B$: for all $f \in \mathscr{G}(\Omega)$,

$$f \in \mathscr{R}\|B \Leftrightarrow I_B f \in \mathscr{R}|B. \tag{10}$$

In our version, updating corresponds to intersecting the cone $\mathscr{R}$ with the linear subspace $\mathscr{G}(\Omega)|B$, which results in a cone $\mathscr{R}|B$ of lower dimension. And since we can uniquely identify a gamble $f = I_B f$ in $\mathscr{G}(\Omega)|B$ with a gamble on B, namely its restriction f_B to B, and *vice versa*, we can also identify $\mathscr{R}|B$ with a set of desirable gambles on B:

$$\mathscr{R} \rfloor B := \{f_B \colon f \in \mathscr{R}|B\} = \{f_B \colon f \in \mathscr{R}\|B\} \subseteq \mathscr{G}(B). \tag{11}$$

Proposition 5. *If $\mathscr{R}$ is a coherent set of desirable gambles on Ω, then $\mathscr{R}|B$ is coherent relative to $\mathscr{G}(\Omega)|B$, or equivalently, $\mathscr{R}\rfloor B$ is a coherent set of desirable gambles on B.*

Our subject takes $\mathscr{R}|B$ (or $\mathscr{R}\rfloor B$) as his set of desirable gambles contingent on observing the event B.

3 Finite exchangeable sequences

Now that we have become better versed in the theory of sets of desirable gambles, we are going to focus on the main topic: reasoning about finite exchangeable sequences. We first show how they are related to count vectors (Sec. 3.1). Then we are ready to give a desirability-based definition of exchangeability (Sec. 3.2) and treat natural extension and updating under exchangeability (Secs. 3.3 and 3.4). After presenting our Finite Representation Theorem (Sec. 3.5), we can show what natural extension and updating under exchangeability look like in terms of the count vector representation (Secs. 3.6 and 3.7).

Consider random variables $X_1, \ldots, X_N$ taking values in a non-empty finite set $\mathscr{X}$,[6] where $N \in \mathbb{N}_0$, i.e., a positive (non-zero) integer. The possibility space is $\Omega = \mathscr{X}^N$.

3.1 Count vectors

We denote by $x = (x_1, \ldots, x_N)$ an arbitrary element of $\mathscr{X}^N$. $\mathscr{P}_N$ is the set of all permutations π of the index set $\{1, \ldots, N\}$. With any such permutation π, we associate a permutation of $\mathscr{X}^N$, also denoted by π, and defined by $(\pi x)_k = x_{\pi(k)}$, or in other words, $\pi(x_1, \ldots, x_N) = (x_{\pi(1)}, \ldots, x_{\pi(N)})$. Similarly, we lift π to a permutation π^t of $\mathscr{G}(\mathscr{X}^N)$ by letting $\pi^t f = f \circ \pi$, so $(\pi^t f)(x) = f(\pi x)$.

[6]A lot of functions and sets introduced below will depend on the set $\mathscr{X}$. We do not indicate this explicitly, not to overburden the notation and because we do not consider different sets of values in this paper.

The permutation invariant atoms $[x] := \{\pi x \colon \pi \in \mathscr{P}_N\}$ are the smallest permutation invariant subsets of $\mathscr{X}^N$. We introduce the *counting map*

$$T^N \colon \mathscr{X}^N \to \mathscr{N}^N \colon x \mapsto T^N(x) \tag{12}$$

where $T^N(x)$ is the $\mathscr{X}$-tuple with components

$$T^N_z(x) := |\{k \in \{1,\dots,N\} \colon x_k = z\}| \quad \text{for all } z \in \mathscr{X}, \tag{13}$$

and the set of possible *count vectors* is given by

$$\mathscr{N}^N := \left\{ m \in \mathbb{N}^{\mathscr{X}} \colon \sum_{x\in\mathscr{X}} m_x = N \right\}. \tag{14}$$

If $m = T^N(x)$, then $[x] = \{y \in \mathscr{X}^N \colon T^N(y) = m\}$, so the atom $[x]$ is completely determined by the count vector m of all its the elements, and is therefore also denoted by $[m]$.

3.2 Defining exchangeability

If a subject assesses that $X_1, \dots, X_N$ are exchangeable, this means that for any gamble f and any permutation π, he finds exchanging $\pi^t f$ for f weakly desirable,[7] because he is indifferent between them [cf. 16, Sec. 4.1.1]. Let

$$\mathscr{D}_{\mathscr{P}_N} := \left\{ f - \pi^t f \colon f \in \mathscr{G}(\mathscr{X}^N) \text{ and } \pi \in \mathscr{P}_N \right\}, \tag{15}$$

then we should have that $\mathscr{D}_{\mathscr{P}_N} \subseteq \mathscr{D}_{\mathscr{R}}$. Before we give useful alternative characterisations of exchangeability, we introduce a few notions that will prove crucial further on.

We begin by defining a special linear transformation ex^N of the linear space of gambles $\mathscr{G}(\mathscr{X}^N)$:

$$\mathrm{ex}^N \colon \mathscr{G}(\mathscr{X}^N) \to \mathscr{G}(\mathscr{X}^N) \colon f \mapsto \mathrm{ex}^N(f) := \frac{1}{N!} \sum_{\pi\in\mathscr{P}_N} \pi^t f. \tag{16}$$

Observe that for all gambles f and all permutations π:

$$\mathrm{ex}^N(\pi^t f) = \mathrm{ex}^N(f) \text{ and } \pi^t\left(\mathrm{ex}^N(f)\right) = \mathrm{ex}^N(f). \tag{17}$$

So $\mathrm{ex}^N(f)$ is permutation invariant and therefore constant on the permutation invariant atoms $[m]$, and it assumes the same value for all gambles that can be related to each other through some permutation. What is the value that $\mathrm{ex}^N(f)$ assumes on $[m]$? It is not difficult to see that

$$\mathrm{ex}^N = \sum_{m\in\mathscr{N}^N} \mathrm{MuHy}^N(\cdot|m) I_{[m]}, \tag{18}$$

where we let

$$\mathrm{MuHy}^N(f|m) := \frac{1}{|[m]|} \sum_{y\in[m]} f(y) \tag{19}$$

$$|[m]| = \binom{N}{m} := \frac{N!}{\prod_{z\in\mathscr{X}} m_z!}. \tag{20}$$

[7] Note that the gambles in $\mathscr{D}_{\mathscr{P}_N}$ cannot be assumed to be desirable, because $\mathscr{D}_{\mathscr{P}_N}$ does not avoid non-positivity.

$\mathrm{MuHy}^N(\cdot|m)$ is the linear expectation operator associated with the uniform distribution on the invariant atom $[m]$. It characterises a *multivariate hyper-geometric distribution* [10, Sec. 39.2], associated with random sampling without replacement from an urn with N balls of types $\mathscr{X}$, whose composition is characterised by the count vector m. If we also observe that $\mathrm{ex}^N \circ \mathrm{ex}^N = \mathrm{ex}^N$, we see that ex^N is the linear *projection operator* of $\mathscr{G}(\mathscr{X}^N)$ to the linear space

$$\mathscr{G}_{\mathscr{P}_N}(\mathscr{X}^N) := \left\{ f \in \mathscr{G}(\mathscr{X}^N) \colon (\forall \pi \in \mathscr{P}_N) \pi^t f = f \right\} \tag{21}$$

of all permutation invariant gambles. We also let

$$\mathscr{D}_{\mathscr{U}_N} := \mathrm{span}(\mathscr{D}_{\mathscr{P}_N}) \tag{22}$$

$$= \left\{ f - \mathrm{ex}^N(f) \colon f \in \mathscr{G}(\mathscr{X}^N) \right\} \tag{23}$$

$$= \left\{ f \in \mathscr{G}(\mathscr{X}^N) \colon \mathrm{ex}^N(f) = 0 \right\}, \tag{24}$$

where 'span' denotes linear span. The linear space $\mathscr{D}_{\mathscr{U}_N}$ is the kernel of the linear projection operator ex^N.

Definition 3 (Exchangeability)**.** *A coherent set $\mathscr{R}$ of desirable gambles on $\mathscr{X}^N$ is called* exchangeable *if any (and hence all) of the following equivalent conditions is (are) satisfied:*

(i) *any gamble in $\mathscr{D}_{\mathscr{P}_N}$ is weakly desirable: $\mathscr{D}_{\mathscr{P}_N} \subseteq \mathscr{D}_{\mathscr{R}}$;*
(ii) *$\mathscr{D}_{\mathscr{P}_N} + \mathscr{R} \subseteq \mathscr{R}$;*
(iii) *any gamble in $\mathscr{D}_{\mathscr{U}_N}$ is weakly desirable: $\mathscr{D}_{\mathscr{U}_N} \subseteq \mathscr{D}_{\mathscr{R}}$;*
(iv) *$\mathscr{D}_{\mathscr{U}_N} + \mathscr{R} \subseteq \mathscr{R}$;*

We call a lower prevision $\underline{P}$ on $\mathscr{G}(\mathscr{X}^N)$ exchangeable *if there is some exchangeable coherent set of desirable gambles $\mathscr{R}$ such that $\underline{P} = \underline{P}_{\mathscr{R}}$.*

The conditions (iii)–(iv) of this definition are quite closely related to the desirability version of a de Finetti-like representation theorem for finite exchangeable sequences in terms of sampling without replacement from an urn. They allow us talk about exchangeability without invoking permutations. This is what we will address in Section 3.5.

A number of useful results follow from this definition:

Proposition 6. *Let $\mathscr{R}$ be a coherent set of desirable gambles. If $\mathscr{R}$ is exchangeable then it is also* permutable*: $\pi^t f \in \mathscr{R}$ for all $f \in \mathscr{R}$ and all $\pi \in \mathscr{P}_N$.*

Proposition 7. *Let $\mathscr{R}$ be a coherent and exchangeable set of desirable gambles. For all gambles f and f' on $\mathscr{X}^N$:*

(i) *$f \in \mathscr{R} \Leftrightarrow \mathrm{ex}^N(f) \in \mathscr{R}$;*
(ii) *If $\mathrm{ex}^N(f) = \mathrm{ex}^N(f')$, then $f \in \mathscr{R} \Leftrightarrow f' \in \mathscr{R}$.*

It follows from this last proposition and Eq. (24) that for any coherent and exchangeable set of desirable gambles $\mathscr{R}$:

$$\mathscr{R} \cap \mathscr{D}_{\mathscr{U}_N} = \emptyset. \tag{25}$$

Theorem 8. *Let $\underline{P}$ be a coherent lower prevision on $\mathscr{G}(\mathscr{X}^N)$. Then the following statements are equivalent:*[8]

[8] This shows that the exchangeability of a lower prevision can also be expressed using marginally desirable gambles [see 14, Sec. 3.1.1].

(i) $\underline{P}$ *is exchangeable;*
(ii) $\underline{P}(f)=\overline{P}(f)=0$ *for all* $f\in\mathscr{D}_{\mathscr{P}_N}$*;*
(iii) $\underline{P}(f)=\overline{P}(f)=0$ *for all* $f\in\mathscr{D}_{\mathscr{U}_N}$*.*

3.3 Exchangeable natural extension

Let us denote the set of all coherent and exchangeable sets of desirable gambles on $\mathscr{X}^N$ by

$$\mathbb{D}_{\mathrm{ex}}(\mathscr{X}^N) := \left\{\mathscr{R}\in\mathbb{D}(\mathscr{X}^N)\colon \mathscr{D}_{\mathscr{U}_N}+\mathscr{R}\subseteq\mathscr{R}\right\}. \quad (26)$$

This set is closed under arbitrary non-empty intersections. We shall see further on in Corollary 11 that it is also non-empty, and therefore has a smallest element.

Suppose our subject has an assessment, or in other words, a set $\mathscr{A}$ of gambles on $\mathscr{X}^N$ that he finds desirable. Then we can ask if there is some coherent and exchangeable set of desirable gambles $\mathscr{R}$ that includes $\mathscr{A}$. In other words, we want a set of desirable gambles $\mathscr{R}$ to satisfy the requirements: (i) $\mathscr{R}$ is coherent; (ii) $\mathscr{A}\subseteq\mathscr{R}$; and (iii) $\mathscr{D}_{\mathscr{U}_N}+\mathscr{R}\subseteq\mathscr{R}$. Clearly, the intersection $\bigcap_{i\in I}\mathscr{R}_i$ of an arbitrary non-empty family of sets of desirable gambles $\mathscr{R}_i$, $i\in I$ that satisfy these requirements, will satisfy these requirements as well. This is the idea behind the following results.

Proposition 9. *We say that a set $\mathscr{A}$ of gambles on $\mathscr{X}^N$* avoids non-positivity under exchangeability *if the set of gambles $[\mathscr{G}_0^+(\mathscr{X}^N)\cup\mathscr{A}]+\mathscr{D}_{\mathscr{U}_N}$ avoids non-positivity. Then: (i) $\emptyset$ avoids non-positivity under exchangeability; and (ii) if $\mathscr{A}$ is non-empty, then $\mathscr{A}$ avoids non-positivity under exchangeability iff $\mathscr{A}+\mathscr{D}_{\mathscr{U}_N}$ avoids non-positivity.*

Theorem 10 (Exchangeable natural extension)**.** *Consider a set $\mathscr{A}$ of gambles on $\mathscr{X}^N$, and define its* exchangeable natural extension *$\mathscr{E}_{\mathrm{ex}}^N(\mathscr{A})$ by*

$$\mathscr{E}_{\mathrm{ex}}^N(\mathscr{A}) := \bigcap\left\{\mathscr{R}\in\mathbb{D}_{\mathrm{ex}}(\mathscr{X}^N)\colon \mathscr{A}\subseteq\mathscr{R}\right\} \quad (27)$$

$$= \mathrm{coni}\left(\mathscr{D}_{\mathscr{U}_N}+[\mathscr{G}_0^+(\mathscr{X}^N)\cup\mathscr{A}]\right) \quad (28)$$

$$= \mathscr{D}_{\mathscr{U}_N}+\mathscr{E}(\mathscr{A}). \quad (29)$$

Then the following statements are equivalent:

(i) $\mathscr{A}$ avoids non-positivity under exchangeability*;*
(ii) *$\mathscr{A}$ is included in some coherent and exchangeable set of desirable gambles;*
(iii) *$\mathscr{E}_{\mathrm{ex}}^N(\mathscr{A})\neq\mathscr{G}(\mathscr{X}^N)$;*
(iv) *$\mathscr{E}_{\mathrm{ex}}^N(\mathscr{A})$ is a coherent and exchangeable set of desirable gambles;*
(v) *$\mathscr{E}_{\mathrm{ex}}^N(\mathscr{A})$ is the smallest coherent and exchangeable set of desirable gambles that includes $\mathscr{A}$.*

Corollary 11. *The set $\mathbb{D}_{\mathrm{ex}}(\mathscr{X}^N)$ is non-empty, and has a smallest element*

$$\mathscr{R}_{\mathrm{ex,v}}^N := \mathscr{E}_{\mathrm{ex}}^N(\emptyset) = \mathscr{D}_{\mathscr{U}_N}+\mathscr{G}_0^+(\mathscr{X}^N). \quad (30)$$

3.4 Updating exchangeable models

Consider an exchangeable and coherent set of desirable gambles $\mathscr{R}$ on $\mathscr{X}^N$, and assume that we have observed the values $\check{x}=(\check{x}_1,\check{x}_2,\dots,\check{x}_{\check{n}})$ of the first $\check{n}$ variables $X_1,\dots,X_{\check{n}}$, and that we want to make inferences about the remaining $\hat{n}:=N-\check{n}$ variables. To do this, we simply update the set $\mathscr{R}$ with the set $C_{\check{x}}=\{\check{x}\}\times\mathscr{X}^{\hat{n}}$, to obtain the set $\mathscr{R}|C_{\check{x}}$, also denoted as $\mathscr{R}|\check{x}=\left\{f\in\mathscr{R}\colon fI_{C_{\check{x}}}=f\right\}$. As we have seen in Section 2.3, this set can be identified with a coherent set of desirable gambles on $\mathscr{X}^{\hat{n}}$, which we denote by $\mathscr{R}\rfloor\check{x}$. With obvious notations:[9]

$$\mathscr{R}\rfloor\check{x} = \left\{f\in\mathscr{G}(\mathscr{X}^{\hat{n}})\colon fI_{C_{\check{x}}}\in\mathscr{R}\right\}. \quad (31)$$

We already know that updating preserves coherence. We now see that this type of updating on an observed sample also preserves exchangeability.

Proposition 12. *Consider $\check{x}\in\mathscr{X}^{\check{n}}$ and a coherent and exchangeable set of desirable gambles $\mathscr{R}$ on $\mathscr{X}^N$. Then $\mathscr{R}\rfloor\check{x}$ is a coherent and exchangeable set of desirable gambles on $\mathscr{X}^{\hat{n}}$.*

We also introduce another type of updating, where we observe a count vector $\check{m}\in\mathscr{N}^{\check{n}}$, and we update the set $\mathscr{R}$ with the set $C_{\check{m}}=[\check{m}]\times\mathscr{X}^{\hat{n}}$, to obtain the set $\mathscr{R}|C_{\check{m}}$, also denoted as $\mathscr{R}|\check{m}=\left\{f\in\mathscr{R}\colon fI_{C_{\check{m}}}=f\right\}$. This set can be identified with a coherent set of desirable gambles on $\mathscr{X}^{\hat{n}}$, which we also denote by $\mathscr{R}\rfloor\check{m}$. With obvious notations:

$$\mathscr{R}\rfloor\check{m} = \left\{f\in\mathscr{G}(\mathscr{X}^{\hat{n}})\colon fI_{C_{\check{m}}}\in\mathscr{R}\right\}. \quad (32)$$

Proposition 13 (Sufficiency of observed count vectors)**.** *Consider $\check{x},\check{y}\in\mathscr{X}^{\check{n}}$ and a coherent and exchangeable set of desirable gambles $\mathscr{R}$ on $\mathscr{X}^N$. If $\check{y}\in[\check{x}]$, or in other words if $T^{\check{n}}(\check{x})=T^{\check{n}}(\check{y})=:\check{m}$, then $\mathscr{R}\rfloor\check{x}=\mathscr{R}\rfloor\check{y}=\mathscr{R}\rfloor\check{m}$.*

3.5 Finite representation

We now introduce the linear map MuHy^N from the linear space $\mathscr{G}(\mathscr{X}^N)$ to the linear space $\mathscr{G}(\mathscr{N}^N)$, as follows:

$$\mathrm{MuHy}^N\colon \mathscr{G}(\mathscr{X}^N)\to\mathscr{G}(\mathscr{N}^N)\colon f\mapsto\mathrm{MuHy}^N(f):=\mathrm{MuHy}^N(f|\cdot), \quad (33)$$

so $\mathrm{MuHy}^N(f)$ is the gamble on $\mathscr{N}^N$ that assumes the value $\mathrm{MuHy}^N(f|m)$ in the count vector $m\in\mathscr{N}^N$. We also define the linear map T^N from the linear space $\mathscr{G}(\mathscr{N}^N)$ to the linear space $\mathscr{G}_{\mathscr{P}_N}(\mathscr{X}^N)$ as follows:

$$\mathrm{T}^N\colon \mathscr{G}(\mathscr{N}^N)\to\mathscr{G}_{\mathscr{P}_N}(\mathscr{X}^N)\colon g\mapsto\mathrm{T}^N(g):=g\circ T^N, \quad (34)$$

so $\mathrm{T}^N(g)$ is the permutation invariant gamble on $\mathscr{X}^N$ that assumes the constant value $g(m)$ on the invariant atom $[m]$. For all $f\in\mathscr{G}(\mathscr{X}^N)$, $\mathrm{ex}^N(f)=\mathrm{T}^N\left(\mathrm{MuHy}^N(f)\right)$, and similarly, for all $g\in\mathscr{G}(\mathscr{N}^N)$, $\mathrm{MuHy}^N\left(\mathrm{T}^N(g)\right)=g$. Hence:

$$\mathrm{ex}^N=\mathrm{T}^N\circ\mathrm{MuHy}^N \text{ and } \mathrm{MuHy}^N\circ\mathrm{T}^N=\mathrm{id}_{\mathscr{G}(\mathscr{N}^N)}. \quad (35)$$

[9] Here and further on we silently use cylindrical extension on gambles, i.e., let them 'depend' on extra variables whose value does not influence the value they take.

If we invoke Eq. (17) we find that

$$\mathrm{MuHy}^N(\pi^t f) = \mathrm{MuHy}^N(f). \tag{36}$$

Also taking into account the linearity of MuHy^N and Eq. (16), this leads to

$$\mathrm{MuHy}^N(\mathrm{ex}^N(f)) = \mathrm{MuHy}^N(f). \tag{37}$$

The relationships between the three important linear maps we have introduced above are clarified by the commutative diagram in Fig. 1.

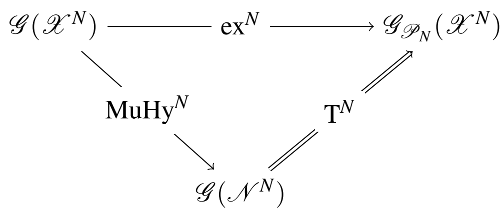


Figure 1: Single sequence length commutative diagram. Double arrows indicate a linear isomorphism.

For every gamble f on $\mathscr{X}^N$, $f = \mathrm{ex}^N(f) + [f - \mathrm{ex}^N(f)]$, so it can be decomposed as a sum of a permutation invariant gamble $\mathrm{ex}^N(f)$ and an element $f - \mathrm{ex}^N(f)$ of the kernel $\mathscr{D}_{\mathscr{U}_N}$ of the linear projection operator ex^N. Since we know that MuHy^N is a linear isomorphism between the spaces $\mathscr{G}_{\mathscr{P}_N}(\mathscr{X}^N)$ and $\mathscr{G}(\mathscr{N}^N)$, we now investigate whether we can represent coherent and exchangeable $\mathscr{R}$ by some set of desirable *count gambles* on $\mathscr{N}^N$.

Theorem 14 (Finite Representation)**.** *A set of desirable gambles $\mathscr{R}$ on $\mathscr{X}^N$ is coherent and exchangeable iff there is some coherent set $\mathscr{S}$ of desirable gambles on $\mathscr{N}^N$ such that*

$$\mathscr{R} = (\mathrm{MuHy}^N)^{-1}(\mathscr{S}), \tag{38}$$

and in that case this $\mathscr{S}$ is uniquely determined by

$$\mathscr{S} = \left\{ g \in \mathscr{G}(\mathscr{N}^N) \colon\ \mathrm{T}^N(g) \in \mathscr{R} \right\} = \mathrm{MuHy}^N(\mathscr{R}). \tag{39}$$

Corollary 15. *A lower prevision $\underline{P}$ on $\mathscr{G}(\mathscr{X}^N)$ is coherent and exchangeable iff there is some coherent lower prevision $\underline{Q}$ on $\mathscr{G}(\mathscr{N}^N)$ such that $\underline{P} = \underline{Q} \circ \mathrm{MuHy}^N$. In that case $\underline{Q}$ is uniquely determined by $\underline{Q} = \underline{P} \circ \mathrm{T}^N$.*

We call the set $\mathscr{S}$ and the lower prevision $\underline{Q}$ the *count representations* of the exchangeable set $\mathscr{R}$ and the exchangeable lower prevision $\underline{P}$, respectively. Our Finite Representation Theorem allows us to give an appealing geometrical interpretation to the notions of exchangeability and representation. The exchangeability of $\mathscr{R}$ means that it is completely determined by its count representation $\mathrm{MuHy}^N(\mathscr{R})$, or what amounts to the same thing since T^N is a linear isomorphism: by its projection $\mathrm{ex}^N(\mathscr{R})$ on the linear space of all permutation invariant gambles. This turns count vectors into useful sufficient statistics (compare with Proposition 13), because the dimension of $\mathscr{G}(\mathscr{N}^N)$ is typically much smaller than that of $\mathscr{G}(\mathscr{X}^N)$.

3.6 Exchangeable natural extension and representation

The exchangeable natural extension is easy to calculate using natural extension in terms of count representations, and the following simple result therefore has important consequences for practical implementations of reasoning and inference under exchangeability.

Theorem 16. *Let $\mathscr{A}$ be a set of gambles on $\mathscr{X}^N$, then*

(i) *$\mathscr{A}$ avoids non-positivity under exchangeability iff $\mathrm{MuHy}^N(\mathscr{A})$ avoids non-positivity.*
(ii) $\mathrm{MuHy}^N(\mathscr{E}^N_{\mathrm{ex}}(\mathscr{A})) = \mathscr{E}(\mathrm{MuHy}^N(\mathscr{A}))$.

3.7 Updating and representation

Suppose, as in Section 3.4, that we update a coherent and exchangeable set of desirable gambles $\mathscr{R}$ after observing a sample $\check{x}$ with count vector $\check{m}$. This leads to an updated coherent and exchangeable set of desirable gambles $\mathscr{R}\rfloor\check{x} = \mathscr{R}\rfloor\check{m}$ on $\mathscr{X}^{\hat{n}}$. Here, we take a closer look at the corresponding set of desirable gambles on $\mathscr{N}^{\hat{n}}$, which we denote (symbolically) by $\mathscr{S}\rfloor\check{m}$ (but we do not want to suggest with this notation that this is in some way an updated set of gambles!). The Finite Representation Theorem 14 tells us that $\mathscr{S}\rfloor\check{m} = \mathrm{MuHy}^{\hat{n}}(\mathscr{R}\rfloor\check{m})$, but is there a direct way to infer the count representation $\mathscr{S}\rfloor\check{m}$ of $\mathscr{R}\rfloor\check{m}$ from the count representation $\mathscr{S} = \mathrm{MuHy}^N(\mathscr{R})$ of $\mathscr{R}$?

To show that there is, we need to introduce two new notions: the *likelihood function*

$$L_{\check{m}} \colon \mathscr{N}^{\hat{n}} \to \mathbb{R} \colon \hat{m} \mapsto L_{\check{m}}(\hat{m}) := \frac{|[\check{m}]|\,|[\hat{m}]|}{|[\check{m}+\hat{m}]|}, \tag{40}$$

associated with sampling without replacement, and the linear map $+_{\check{m}}$ from the linear space $\mathscr{G}(\mathscr{N}^{\hat{n}})$ to the linear space $\mathscr{G}(\mathscr{N}^N)$ given by

$$+_{\check{m}} \colon \mathscr{G}(\mathscr{N}^{\hat{n}}) \to \mathscr{G}(\mathscr{N}^N) \colon g \mapsto +_{\check{m}} g \tag{41}$$

where

$$+_{\check{m}}\, g(M) = \begin{cases} g(M - \check{m}) & \text{if } M \geq \check{m} \\ 0 & \text{otherwise.} \end{cases} \tag{42}$$

Proposition 17. *Consider a coherent and exchangeable set of desirable gambles $\mathscr{R}$ on $\mathscr{X}^N$, with count representation $\mathscr{S}$. Let $\mathscr{S}\rfloor\check{m}$ be the count representation of the coherent and exchangeable set of desirable gambles $\mathscr{R}\rfloor\check{m}$, obtained after updating $\mathscr{R}$ with a sample $\check{x}$ with count vector $\check{m}$. Then*

$$\mathscr{S}\rfloor\check{m} = \left\{ g \in \mathscr{G}(\mathscr{N}^{\hat{n}}) \colon\ +_{\check{m}}(L_{\check{m}} g) \in \mathscr{S} \right\}. \tag{43}$$

4 Conclusions

We have shown that modelling an exchangeability assessment using sets of desirable gambles is not only possible, but also elegant.

Our results indicate that, using sets of desirable gambles, it is conceptually easy to reason about exchangeable sequences. Calculating the natural extension and updating are but simple geometrical operations: taking unions, sums and conical hulls and taking intersections, respectively. This approach has the added advantage that the exchangeability assessment is preserved under updating, also when the conditioning event has lower probability zero, which does not hold when using (lower) previsions (although this might be remedied by using full conditional measures).

Moreover, using our Finite Representation Theorem, reasoning about exchangeable sequences can be reduced to reasoning about count vectors. Working with this representation automatically guarantees that exchangeability is satisfied. The representation for the natural extension and for updated models can be derived directly from the representation of the original model, without having to go back to the (more complex) world of sequences. We have also looked at the problem of representation for infinite sequences, but will report this elsewhere.

The conceptual techniques employed in this paper are not restricted in use to a treatment of exchangeability. They could be applied to other structural assessments, e.g., invariance assessments, as long as this assessment allows us to identify a characterising set of weakly desirable gambles that is sufficiently well-behaved (cf. the first paragraph of Sec. 3.2). This idea was briefly taken up by one of us in another paper [1], but clearly merits further attention.

Thinking in even broader terms, we feel that using sets of desirable gambles can provide a refreshing and fruitful approach to many problems in uncertainty modelling, not only those related to structural assessments.

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A Proofs

We provide proofs for the more involved results.

Proof of Proposition 4. Since it follows from Theorem 3 that $\underline{P}_{\mathscr{R}}(f-\underline{P}_{\mathscr{R}}(f))=\underline{P}_{\mathscr{R}}(f)-\underline{P}_{\mathscr{R}}(f)=0$ for all gambles f, it follows that $\mathscr{M}_{\mathscr{R}}\subseteq\{f\in\mathscr{G}(\Omega)\colon \underline{P}_{\mathscr{R}}(f)=0\}$. For the converse inequality, assume that $\underline{P}_{\mathscr{R}}(f)=0$ holds; then $f=f-\underline{P}_{\mathscr{R}}(f)\in\mathscr{M}_{\mathscr{R}}$.

This also means that $\underline{P}_{\mathscr{R}}(g)=0$ iff $g\in\mathscr{M}_{\mathscr{R}}$, so for every gamble f we can write:

$$\underline{P}_{\mathscr{M}_{\mathscr{R}}}(f)=\sup\{\mu\in\mathbb{R}\colon f-\mu\in\mathscr{M}_{\mathscr{R}}\} \tag{44}$$

$$=\sup\{\mu\in\mathbb{R}\colon \underline{P}_{\mathscr{R}}(f-\mu)=0\} \tag{45}$$

$$=\sup\{\mu\in\mathbb{R}\colon \mu=\underline{P}_{\mathscr{R}}(f)\}=\underline{P}_{\mathscr{R}}(f), \tag{46}$$

which proves the equality of $\underline{P}_{\mathscr{M}_{\mathscr{R}}}$ and $\underline{P}_{\mathscr{R}}$. □

Proof of Proposition 5. We need to prove that D1–D4 hold for $\mathscr{R}|B$. For D1, consider $f\in\mathscr{G}(\Omega)|B$ and assume that $f=0$. Then by coherence $f\notin\mathscr{R}$ and hence $f\notin\mathscr{R}|B$. For D2, consider $f\in\mathscr{G}(\Omega)|B$ and assume that $f>0$. Then by coherence $f\in\mathscr{R}$ and hence $f\in\mathscr{R}|B$. The proof for D3 is similar to the one for D4. For D4, consider $f_1,f_2\in\mathscr{R}|B$, then on the one hand $f_1,f_2\in\mathscr{R}$ and therefore $f_1+f_2\in\mathscr{R}$ by coherence; and on the other hand $f_1,f_2\in\mathscr{G}(\Omega)|B$ and therefore $f_1+f_2=I_Bf_1+I_Bf_2=I_B(f_1+f_2)$, so $f_1+f_2\in\mathscr{G}(\Omega)|B$ and hence $f_1+f_2\in\mathscr{R}|B$. □

Proof of the equivalences in Definition 3. That (i)⇔(ii) and (iii)⇔(iv) is an immediate consequence of the definition of weak desirability. We continue to show that (i)⇔(iii). For the '⇒' part, observe that $f-\mathrm{ex}^N(f)=\frac{1}{N!}\sum_{\pi\in\mathscr{P}_N}[f-\pi^tf]\in\mathscr{D}_{\mathscr{R}}$, since $\mathscr{D}_{\mathscr{R}}$ is a convex cone by Proposition 2. For the '⇐' part, consider any $f\in\mathscr{G}(\mathscr{X}^N)$ and $\pi\in\mathscr{P}_N$. Consider any $f'\in\mathscr{R}$. Then by assumption both $f-\mathrm{ex}^N(f)+f'/2$ and $\pi^t(-f)-\mathrm{ex}^N(\pi^t(-f))+f'/2$ belong to $\mathscr{R}$. Hence, because $\mathscr{R}$ is closed under addition, their sum $f-\pi^tf+f'$, obtained using Eq. (17), also belongs to $\mathscr{R}$. Hence $f-\pi^tf$ is weakly desirable. □

Proof of Proposition 6. Consider $f\in\mathscr{R}$. Since $\pi^tf-f=(-f)-\pi^t(-f)\in\mathscr{D}_{\mathscr{P}_N}$, we see that $\pi^tf=f+\pi^tf-f\in\mathscr{R}+\mathscr{D}_{\mathscr{P}_N}\subseteq\mathscr{R}$, using the exchangeability condition of Def. 3(ii). □

Proof of Proposition 7. The first statement is a consequence of the second, with $f'=\mathrm{ex}^N(f)$, because then $\mathrm{ex}^N(f')=\mathrm{ex}^N(\mathrm{ex}^N(f))=\mathrm{ex}^N(f)$. For the second statement, consider arbitrary gambles f and f' on $\mathscr{X}^N$ such that $\mathrm{ex}^N(f)=\mathrm{ex}^N(f')$, and assume that $f\in\mathscr{R}$. We prove that then also $f'\in\mathscr{R}$. Since $\mathrm{ex}^N(f)-f=(-f)-\mathrm{ex}^N(-f)\in\mathscr{D}_{\mathscr{R}}$ and $f'-\mathrm{ex}^N(f')\in\mathscr{D}_{\mathscr{R}}$, we see that $f'-f\in\mathscr{D}_{\mathscr{R}}$ by WD4, and therefore $f'=f+f'-f\in\mathscr{R}+\mathscr{D}_{\mathscr{R}}\subseteq\mathscr{R}$. □

Proof of Theorem 8. We give a circular proof. We first show that (ii) holds if $\underline{P}$ is exchangeable, i.e., if there is some coherent and exchangeable $\mathscr{R}$ such that $\underline{P}=\underline{P}_{\mathscr{R}}$. We already know from Theorem 3 that $\underline{P}=\underline{P}_{\mathscr{R}}$ satisfies P1–P3, because $\mathscr{R}$ is coherent. Consider any $f\in\mathscr{D}_{\mathscr{P}_N}$. Since $\mathscr{D}_{\mathscr{P}_N}\subseteq\mathscr{D}_{\mathscr{R}}$, it also follows from Theorem 3 that $\underline{P}_{\mathscr{R}}(f)\geq 0$ and similarly $-\overline{P}_{\mathscr{R}}(f)=\underline{P}_{\mathscr{R}}(-f)\geq 0$ because also $-f\in\mathscr{D}_{\mathscr{P}_N}$. Hence indeed $0\leq\underline{P}_{\mathscr{R}}(f)\leq\overline{P}_{\mathscr{R}}(f)\leq 0$, where the second inequality is a consequence of P1 and P2.

That (ii) implies (iii) follows the super-additivity of $\underline{P}$ and the sub-additivity of $\overline{P}$.

Finally, we show that (iii) implies that $\underline{P}$ is exchangeable. The standard argument in [17, Section 6] tells us that $\mathscr{R}':=\big\{f\in\mathscr{G}(\mathscr{X}^N)\colon f>0 \text{ or } \underline{P}(f)>0\big\}$ is a coherent set of desirable gambles such that $\underline{P}_{\mathscr{R}'}=\underline{P}$. Now consider the set $\mathscr{R}:=\mathscr{R}'+\mathscr{D}_{\mathscr{U}_N}$. We show that this $\mathscr{R}$ is a coherent and exchangeable set of desirable gambles, and that $\underline{P}_{\mathscr{R}}=\underline{P}$. It is clear from its definition that $\mathscr{R}$ satisfies D2, D3 and D4, so let us assume *ex absurdo* that $0\in\mathscr{R}$, meaning that there is some $f\in\mathscr{R}'$ such that $f':=-f\in\mathscr{D}_{\mathscr{U}_N}$. There are two possibilities. Either $f>0$, so $f'<0$, which contradicts Lemma 18. Or $\underline{P}(f)>0$. But it follows from the coherence of the lower prevision $\underline{P}$ and the assumption that $0=\underline{P}(f+f')=\underline{P}(f)>0$, a contradiction too. So $\mathscr{R}$ satisfies D1 as well, and is therefore coherent. It is obvious that $\mathscr{R}$ is exchangeable: $\mathscr{R}+\mathscr{D}_{\mathscr{U}_N}=\mathscr{R}'+\mathscr{D}_{\mathscr{U}_N}+\mathscr{D}_{\mathscr{U}_N}=\mathscr{R}'+\mathscr{D}_{\mathscr{U}_N}=\mathscr{R}$. The proof is complete if we can show that $\underline{P}=\underline{P}_{\mathscr{R}}$. Fix any gamble f. Observe that $f-\alpha\in\mathscr{R}$ iff there are $f'\in\mathscr{R}$ and $f''\in\mathscr{D}_{\mathscr{U}_N}$ such that $f-\alpha=f'+f''$. But then it follows from the coherence of $\underline{P}$ and the assumption that $\underline{P}(f)=\alpha+\underline{P}(f'+f'')=\alpha+\underline{P}(f')\geq\alpha$, and therefore $\underline{P}_{\mathscr{R}}(f)\leq\underline{P}(f)=\underline{P}_{\mathscr{R}'}(f)$. For the converse inequality, we infer from $0 \in \mathscr{D}_{\mathscr{U}_N}$ that $\mathscr{R}' \subseteq \mathscr{R}$, and therefore $\underline{P}_{\mathscr{R}'} \leq \underline{P}_{\mathscr{R}}$. □

Lemma 18. *For all f in $\mathscr{D}_{\mathscr{U}_N}$, $f \not< 0$.*

Proof. First of all, observe that for any gamble f' on $\mathscr{X}^N$, if $f' > 0$ then also $\mathrm{ex}^N(f') > 0$. Now consider $f \in \mathscr{D}_{\mathscr{U}_N}$ and assume *ex absurdo* that $f < 0$. Then $-f > 0$ and therefore $-\mathrm{ex}^N(f) = \mathrm{ex}^N(-f) > 0$, whence $\mathrm{ex}^N(f) < 0$. But since $f \in \mathscr{D}_{\mathscr{U}_N}$ we also have that $\mathrm{ex}^N(f) = 0$, a contradiction. □

Proof of Proposition 9. For the first statement, we have to prove that $\mathscr{G}_0^+(\mathscr{X}^N) + \mathscr{D}_{\mathscr{U}_N}$ avoids non-positivity. Consider any $f' \in \mathscr{D}_{\mathscr{U}_N}$ and any $f'' \in \mathscr{G}_0^+(\mathscr{X}^N)$, then we have to prove that $f := f' + f'' \not\leq 0$. There are two possibilities. Either $f' = 0$ and then $f = f'' > 0$. Or $f' \neq 0$, and then Lemma 18 tells us that $f' \not< 0$, whence $f' \not\leq 0$ and therefore *a fortiori* $f \not\leq 0$.

For the second statement, it clearly suffices to prove the 'if' part. Assume therefore that $\mathscr{A} + \mathscr{D}_{\mathscr{U}_N}$ avoids non-positivity. Consider any f in $\mathrm{coni}([\mathscr{G}_0^+(\mathscr{X}^N) \cup \mathscr{A}] + \mathscr{D}_{\mathscr{U}_N})$, so there are $n \geq 1$, $\lambda_k \in \mathbb{R}^+$, $f' \in \mathscr{D}_{\mathscr{U}_N}$, $f_k \in \mathscr{G}_0^+(\mathscr{X}^N) \cup \mathscr{A}$ such that $f = f' + \sum_{k=1}^n \lambda_k f_k$. Let $I := \{k \in \{1,\dots,n\} \colon f_k > 0\}$ then $f_\ell \in \mathscr{A}$ for all $\ell \notin I$. By assumption $f' + \sum_{\ell \notin I} \lambda_\ell f_\ell \not\leq 0$, and therefore *a fortiori* $f \not\leq 0$. □

Proof of Theorem 10. It is immediately clear from the fact that $\mathbb{D}_{\mathrm{ex}}(\mathscr{X}^N)$ is closed under arbitrary non-empty intersections, the definition of $\mathscr{E}_{\mathrm{ex}}^N(\mathscr{A})$, and the fact that $\mathscr{G}(\mathscr{X}^N)$ is not a coherent set of desirable gambles, that the last four statements are equivalent. We now prove (i)⇔(ii).

First, assume that $\mathscr{A}$, and therefore also $\mathscr{G}_0^+(\mathscr{X}^N) \cup \mathscr{A}$, is included in some coherent and exchangeable set of desirable gambles $\mathscr{R}$. By exchangeability, $[\mathscr{G}_0^+(\mathscr{X}^N) \cup \mathscr{A}] + \mathscr{D}_{\mathscr{U}_N} \subseteq \mathscr{R} + \mathscr{D}_{\mathscr{U}_N} \subseteq \mathscr{R}$. Since $\mathrm{coni}(\mathscr{R}) = \mathscr{R}$ avoids non-positivity, so does any of its subsets, and therefore in particular $[\mathscr{G}_0^+(\mathscr{X}^N) \cup \mathscr{A}] + \mathscr{D}_{\mathscr{U}_N}$. This means that $\mathscr{A}$ indeed avoids non-positivity under exchangeability.

Conversely, assume that $\mathscr{A}$ avoids non-positivity under exchangeability. For the sake of convenience, denote the set on the right-hand side of Eq. (28) by $\mathscr{R}^*$. It is clear that $\mathscr{R}^*$ satisfies D2, D3 and D4. Consider any $f \in \mathscr{R}^*$, then $f \not\leq 0$, precisely because $\mathscr{A}$ avoids non-positivity under exchangeability. Hence $\mathscr{R}^*$ also satisfies D1, and is therefore coherent. To show that $\mathscr{R}^*$ is exchangeable, again consider any $f \in \mathscr{R}^*$, so there are $n \geq 1$, $\lambda_k \in \mathbb{R}^+$, $f' \in \mathscr{D}_{\mathscr{U}_N}$, $f_k \in \mathscr{G}_0^+(\mathscr{X}^N) \cup \mathscr{A}$ such that $f = f' + \sum_{k=1}^n \lambda_k f_k$. Then for any $f'' \in \mathscr{D}_{\mathscr{U}_N}$ we see that $f' + f'' \in \mathscr{D}_{\mathscr{U}_N}$ and therefore indeed $f + f'' = (f' + f'') + \sum_{k=1}^n \lambda_k f_k \in \mathscr{R}^*$.

Since $\mathscr{A} \subseteq \mathscr{R}^*$, the proof of the equivalences is complete. We now turn to the proof of Eq. (28), i.e., we prove that $\mathscr{E}_{\mathrm{ex}}^N(\mathscr{A}) = \mathscr{R}^*$. It is clear that any coherent and exchangeable set of desirable gambles that includes $\mathscr{A}$, must also include $\mathscr{R}^*$, by the axioms D2, D3, and D4. Since we have just proved that $\mathscr{R}^*$ is coherent and exchangeable, it is the smallest coherent and exchangeable set of desirable gambles that includes $\mathscr{A}$. The desired equality now follows because we have assumed that (i) holds, and we have just proved that (i) implies (v).

Eq. (29) follows from Eq. (28) and Theorem 1, since $\mathscr{D}_{\mathscr{U}_N}$ is a cone. □

Proof of Corollary 11. This is an immediate consequence of Proposition 9(i) and Theorem 10. □

Proof of Proposition 12. The coherence of $\mathscr{R}\rfloor\check{x}$ is guaranteed by Proposition 5. We show that $\mathscr{R}\rfloor\check{x}$ is exchangeable. Consider arbitrary $f \in \mathscr{G}(\mathscr{X}^{\hat{n}})$, $\hat{\pi} \in \mathscr{P}_{\hat{n}}$ and $f_1 \in \mathscr{R}\rfloor\check{x}$. Then we must show that $f_1 + f - \hat{\pi}^t f \in \mathscr{R}\rfloor\check{x}$, or in other words that $I_{C_{\check{x}}}[f_1 + f - \hat{\pi}^t f] \in \mathscr{R}$. But since $f_1 \in \mathscr{R}\rfloor\check{x}$, we know that $I_{C_{\check{x}}} f_1 \in \mathscr{R}$. And if we consider the permutation $\pi \in \mathscr{P}_N$ defined by

$$\pi(k) := \begin{cases} k & 1 \leq k \leq \check{n} \\ \check{n} + \hat{\pi}(k - \check{n}) & \check{n}+1 \leq k \leq N, \end{cases} \tag{47}$$

then clearly $I_{C_{\check{x}}} \hat{\pi}^t f = \pi^t(I_{C_{\check{x}}} f)$ and therefore $I_{C_{\check{x}}}[f_1 + f - \hat{\pi}^t f] = I_{C_{\check{x}}} f_1 + I_{C_{\check{x}}} f - \pi^t(I_{C_{\check{x}}} f)$ and this gamble belongs to $\mathscr{R}$ because $\mathscr{R}$ is exchangeable. □

Proof of Proposition 13. Consider $\check{\pi} \in \mathscr{P}_{\check{n}}$ and any gamble f on $\mathscr{X}^{\hat{n}}$. Assume that $I_{C_{\check{x}}} f \in \mathscr{R}$.

We first prove that $I_{C_{\check{\pi}\check{x}}} f \in \mathscr{R}$. Consider the permutation $\pi \in \mathscr{P}_N$ defined by

$$\pi(k) := \begin{cases} \check{\pi}^{-1}(k) & 1 \leq k \leq \check{n} \\ k & \check{n}+1 \leq k \leq N, \end{cases} \tag{48}$$

then clearly $\pi^t(I_{C_{\check{x}}} f) = (I_{C_{\check{x}}} f) \circ \pi = (I_{C_{\check{x}}} \circ \check{\pi}^{-1}) f = I_{C_{\check{\pi}\check{x}}} f$, so it follows from Proposition 6 that indeed $I_{C_{\check{\pi}\check{x}}} f \in \mathscr{R}$. This already implies that $\mathscr{R}\rfloor\check{x} = \mathscr{R}\rfloor\check{\pi}\check{x}$, and therefore also that $\mathscr{R}\rfloor\check{x} = \mathscr{R}\rfloor\check{y}$.

Since $\mathscr{R}$ is coherent, it also follows from $I_{C_{\check{x}}} f \in \mathscr{R}$ and the reasoning above that $I_{C_{\check{m}}} f = \sum_{\check{y} \in [\check{m}]} I_{C_{\check{y}}} f \in \mathscr{R}$, whence $\mathscr{R}\rfloor\check{x} \subseteq \mathscr{R}\rfloor\check{m}$. To prove the converse inequality, assume that $I_{C_{\check{m}}} f \in \mathscr{R}$. We know that $[\check{m}] = \{\check{\pi}\check{x} \colon \check{\pi} \in \mathscr{P}_{\check{n}}\}$, and therefore for any $\check{y} \in [\check{m}]$ we can pick a $\check{\pi}_{\check{y}} \in \mathscr{P}_{\check{n}}$ such that $\check{\pi}_{\check{y}} \check{x} = \check{y}$. With this $\check{\pi}_{\check{y}}$ we construct a permutation $\pi_{\check{y}} \in \mathscr{P}_N$ in the manner described above, which satisfies $\pi_{\check{y}}^t(I_{C_{\check{x}}} f) = I_{C_{\check{y}}} f$. But then the exchangeability and coherence of $\mathscr{R}$ tell us that

$$\begin{aligned} I_{C_{\check{m}}} f + \sum_{\check{y} \in [\check{m}]} [(I_{C_{\check{x}}} f) - \pi_{\check{y}}^t(I_{C_{\check{x}}} f)] &= I_{C_{\check{m}}} f + f \sum_{\check{y} \in [\check{m}]} [I_{C_{\check{x}}} - I_{C_{\check{y}}}] \\ &= |[\check{m}]| f I_{C_{\check{x}}} \end{aligned} \tag{49}$$

belongs to $\mathscr{R}$, whence also $I_{C_{\check{x}}} f \in \mathscr{R}$, by coherence. □

Proof of Theorem 14. We begin with the sufficiency part. Assume that there is some coherent set $\mathscr{S}$ of desirable gambles on $\mathscr{N}^N$ such that $\mathscr{R} = (\mathrm{MuHy}^N)^{-1}(\mathscr{S})$. We show that $\mathscr{R}$ is coherent and exchangeable, and that $\mathscr{S} = \mathrm{MuHy}^N(\mathscr{R})$.

We first show that $\mathscr{R}$ is coherent. For D1, consider $f \in \mathscr{G}(\mathscr{X}^N)$ with $f = 0$. Then obviously also $\mathrm{MuHy}^N(f) = 0$ and therefore $\mathrm{MuHy}^N(f) \notin \mathscr{S}$. Hence $f \notin \mathscr{R}$. For D2, let $f > 0$. Then obviously also $\mathrm{MuHy}^N(f) > 0$, and therefore $\mathrm{MuHy}^N(f) \in \mathscr{S}$. Hence $f \in \mathscr{R}$. The proof for D3 is similar to the one for D4. For D4, let $f_1, f_2 \in \mathscr{R}$. Then $g_1 := \mathrm{MuHy}^N(f_1) \in \mathscr{S}$ and $g_2 := \mathrm{MuHy}^N(f_2) \in \mathscr{S}$. This implies that $\mathrm{MuHy}^N(f_1 + f_2) = g_1 + g_2 \in \mathscr{S}$, so again $f_1 + f_2 \in \mathscr{R}$.

To show that $\mathscr{R}$ is exchangeable, consider any $f \in \mathscr{R}$ and $f' \in \mathscr{D}_{\mathscr{U}_N}$. We have to show that $f + f' \in \mathscr{R}$. It is clear that $\mathrm{MuHy}^N(f + f') = \mathrm{MuHy}^N(f) + 0 = \mathrm{MuHy}^N(f) \in \mathscr{S}$. Hence $f + f' \in (\mathrm{MuHy}^N)^{-1}(\mathscr{S})$, so indeed $f + f' \in \mathscr{R}$.

We show that $\mathscr{S} = \mathrm{MuHy}^N(\mathscr{R})$. Consider any $g \in \mathscr{G}(\mathscr{N}^N)$, then using Eq. (35), $\mathrm{MuHy}^N(\mathrm{T}^N(g)) = g$. Since by assumption $\mathscr{R} = (\mathrm{MuHy}^N)^{-1}(\mathscr{S})$, we see that

$$g \in \mathscr{S} \Leftrightarrow \mathrm{MuHy}^N(\mathrm{T}^N(g)) \in \mathscr{S} \Leftrightarrow \mathrm{T}^N(g) \in \mathscr{R}. \quad (50)$$

This shows that $\mathscr{S} = \{g \in \mathscr{G}(\mathscr{N}^N) \colon \mathrm{T}^N(g) \in \mathscr{R}\}$. We show that also $\mathscr{S} = \mathrm{MuHy}^N(\mathscr{R})$. Let $g \in \mathscr{S}$, then we have just proved that $\mathrm{T}^N(g) \in \mathscr{R}$, and therefore, using Eq. (35), $g = \mathrm{MuHy}^N(\mathrm{T}^N(g)) \in \mathrm{MuHy}^N(\mathscr{R})$. Conversely, let $g \in \mathrm{MuHy}^N(\mathscr{R})$. Then there is some $f \in \mathscr{R}$ such that $g = \mathrm{MuHy}^N(f)$ and therefore $\mathrm{T}^N(g) = \mathrm{T}^N(\mathrm{MuHy}^N(f)) = \mathrm{ex}^N(f)$, where the last equality follows from Eq. (35). Now Proposition 7 tells us that $\mathrm{ex}^N(f) \in \mathscr{R}$, because $f \in \mathscr{R}$ and $\mathscr{R}$ is exchangeable. Hence $\mathrm{T}^N(g) \in \mathscr{R}$ and therefore $g \in \mathscr{S}$.

Next, we turn to the necessity part. Suppose that $\mathscr{R}$ is coherent and exchangeable. It suffices to prove that $\mathscr{S} := \mathrm{MuHy}^N(\mathscr{R})$ is a coherent set of desirable gambles on $\mathscr{N}^N$, and that Eq. (38) is satisfied for this choice of $\mathscr{S}$.

We begin with the coherence of $\mathrm{MuHy}^N(\mathscr{R})$. For D1, consider $g \in \mathscr{G}(\mathscr{N}^N)$ with $g = 0$. Assume *ex absurdo* that $g \in \mathrm{MuHy}^N(\mathscr{R})$, meaning that there is some $f \in \mathscr{R}$ such that $0 = g = \mathrm{MuHy}^N(f)$, or in other words $f \in \mathscr{D}_{\mathscr{U}_N}$. This is impossible, due to Eq. (25). For D2, let $g \geq 0$. Then obviously also $f := \mathrm{T}^N(g) \geq 0$. Therefore $f \in \mathscr{R}$ and, because of Eq. (35), $g = \mathrm{MuHy}^N(\mathrm{T}^N(g)) = \mathrm{MuHy}^N(f) \in \mathrm{MuHy}^N(\mathscr{R})$. The proof for D3 is similar to the one for D4. For D4, let $g_1, g_2 \in \mathrm{MuHy}^N(\mathscr{R})$, so there are $f_1, f_2 \in \mathscr{R}$ such that $g_1 = \mathrm{MuHy}^N(f_1)$ and $g_2 = \mathrm{MuHy}^N(f_2)$. Then by coherence of $\mathscr{R}$, $f_1 + f_2 \in \mathscr{R}$, and therefore, by linearity of MuHy^N,

$$\begin{aligned} g_1 + g_2 &= \mathrm{MuHy}^N(f_1) + \mathrm{MuHy}^N(f_2) \\ &= \mathrm{MuHy}^N(f_1 + f_2) \in \mathrm{MuHy}^N(\mathscr{R}). \end{aligned} \quad (51)$$

Finally, we show that $\mathscr{R} = (\mathrm{MuHy}^N)^{-1}(\mathrm{MuHy}^N(\mathscr{R}))$. Consider $f \in \mathscr{R}$, then $\mathrm{MuHy}^N(f) \in \mathrm{MuHy}^N(\mathscr{R})$ and therefore $f \in (\mathrm{MuHy}^N)^{-1}(\mathrm{MuHy}^N(\mathscr{R}))$. Conversely, consider f in $(\mathrm{MuHy}^N)^{-1}(\mathrm{MuHy}^N(\mathscr{R}))$. Then $g := \mathrm{MuHy}^N(f) \in \mathrm{MuHy}^N(\mathscr{R})$, so we infer that there is some $f' \in \mathscr{R}$ such that $g = \mathrm{MuHy}^N(f) = \mathrm{MuHy}^N(f')$. Hence $\mathrm{MuHy}^N(f - f') = 0$, so $f - f' \in \mathscr{D}_{\mathscr{U}_N}$ and therefore $f = f' + f - f' \in \mathscr{R} + \mathscr{D}_{\mathscr{U}_N}$. This implies that $f \in \mathscr{R}$, since $\mathscr{R}$ is exchangeable. □

Proof of Corollary 15. This result can be easily proved as an immediate consequence of Theorem 14 and Eq. (4). As an illustration, we give a more direct proof of the necessity part, based on Theorem 8. This theorem, together with Eq. (35), tells us that for any gamble f on $\mathscr{X}^N$, $\underline{P}(f) = \underline{P}(\mathrm{ex}^N(f)) = \underline{P}(\mathrm{T}^N(\mathrm{MuHy}^N(f))) = \underline{Q}(\mathrm{MuHy}^N(f))$. □

Proof of Theorem 16. We begin with the second statement. Recall that $\mathscr{E}^N_{\mathrm{ex}}(\mathscr{A}) = \mathscr{D}_{\mathscr{U}_N} + \mathscr{E}^N_{\mathrm{ex}}(\mathscr{A})$ from Theorem 10. Since MuHy^N is a linear operator, it commutes with the coni operator, and therefore:

$$\begin{aligned} &\mathrm{MuHy}^N(\mathscr{E}^N_{\mathrm{ex}}(\mathscr{A})) \\ &\quad = \mathrm{MuHy}^N(\mathscr{D}_{\mathscr{U}_N}) + \mathrm{MuHy}^N(\mathscr{E}^N_{\mathrm{ex}}(\mathscr{A})) \\ &\quad = \mathrm{MuHy}^N(\mathscr{E}^N_{\mathrm{ex}}(\mathscr{A})) \\ &\quad = \mathrm{coni}(\mathrm{MuHy}^N(\mathscr{G}^+_0(\mathscr{X}^N) \cup \mathscr{A})) \\ &\quad = \mathrm{coni}(\mathrm{MuHy}^N(\mathscr{G}^+_0(\mathscr{X}^N)) \cup \mathrm{MuHy}^N(\mathscr{A})) \\ &\quad = \mathrm{coni}(\mathscr{G}^+_0(\mathscr{N}^N) \cup \mathrm{MuHy}^N(\mathscr{A})) \\ &\quad = \mathscr{E}(\mathrm{MuHy}^N(\mathscr{A})), \end{aligned}$$

where the second equality follows from $\mathrm{MuHy}^N(\mathscr{D}_{\mathscr{U}_N}) = \{0\}$, the third from Theorem 10, and the last from Theorem 1. The first statement is an immediate consequence of the second and Theorems 1, 10 and 14. □

Proof of Proposition 17. Recall that $g \in \mathscr{S}\rfloor\check{m}$ iff there is some $f \in \mathscr{G}(\mathscr{X}^{\hat{n}})$ such that at the same time $g = \mathrm{MuHy}^{\hat{n}}(f)$ and $I_{C_{[\check{m}]}} f \in \mathscr{R}$, or in other words $\mathrm{MuHy}^N(I_{C_{[\check{m}]}} f) \in \mathscr{S}$. We therefore consider $M \in \mathscr{N}^N$ and observe that

$$\mathrm{MuHy}^N(I_{C_{[\check{m}]}} f | M) = \frac{1}{|[M]|} \sum_{x \in [M]} (I_{C_{[\check{m}]}} f)(x) \quad (52)$$

$$= \frac{1}{|[M]|} \sum_{\substack{\check{x} \in [\check{m}], \hat{x} \in \mathscr{X}^{\hat{n}} \\ (\check{x}, \hat{x}) \in [M]}} f(\hat{x}), \quad (53)$$

so this value is zero unless $M \geq \check{m}$. In that case we can write $M = \check{m} + \hat{m}$, where $\hat{m} := M - \check{m}$ is a count vector in $\mathscr{N}^{\hat{n}}$; so we find that

$$\mathrm{MuHy}^N(I_{C_{[\check{m}]}} f | \check{m} + \hat{m}) = \frac{1}{|[\check{m} + \hat{m}]|} \sum_{\check{x} \in [\check{m}], \hat{x} \in [\hat{m}]} f(\hat{x}) \quad (54)$$

$$= \frac{|[\check{m}]|\,|[\hat{m}]|}{|[\check{m} + \hat{m}]|} \mathrm{MuHy}^{\hat{n}}(f | \hat{m}). \quad (55)$$

Hence indeed $g \in \mathscr{S}\rfloor\check{m}$ iff $+_{\check{m}}(L_{\check{m}} g) \in \mathscr{S}$. □