Introduction to the theory of imprecise probability

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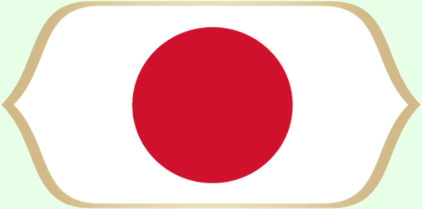
Why would you want your probability to be imprecise?

versus

Uncertainty about outcome of...

versus



Agents (Gamblers)

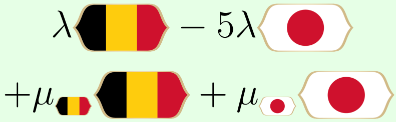
Wiske

Yoko Tsuno

Assessment (gambles accepted)	**Agents (Gamblers)**
1 if  $-\,5$ if 	Wiske
-4 if  $+\,1$ if 	Yoko Tsuno

Natural extension ($\lambda, \mu \geq 0$)

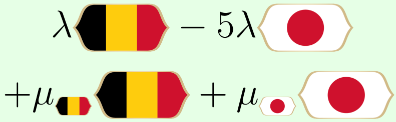
Rational agents



Wiske

Yoko Tsuno

Natural extension $(\lambda, \mu \geq 0)$



COHERENCE

Rational agents

Wiske

Yoko Tsuno

Assessment (gambles accepted) **Agents (Gamblers)**

 $- 5$ ,

-4  $+$ 

Heroine pool

Natural extension ($\lambda, \mu \geq 0$)

Irrational agents

$(\lambda_{\mathsf{W}} - 4\lambda_{\mathsf{Y}} + \mu_{$  $})$ 

$+ (-5\lambda_{\mathsf{W}} + \lambda_{\mathsf{Y}} + \mu_{$  $})$ 

Heroine pool

Natural extension ($\lambda, \mu \geq 0$)

$(\lambda_{\mathsf{W}} - 4\lambda_{\mathsf{Y}} + \mu$$)$

$+ (-5\lambda_{\mathsf{W}} + \lambda_{\mathsf{Y}} + \mu$$)$

SURE LOSS!

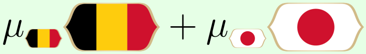
Irrational agents

Heroine pool

Assessment (gambles accepted)	**Agents (Gamblers)**
$\emptyset$	Heroine pool

Natural extension ($\lambda, \mu \geq 0$)

Rational agents



Heroine pool

Natural extension ($\lambda, \mu \geq 0$)

Rational agents

μ   $+ \mu$  

Heroine pool

VACUOUS

Basic concepts

- *Agent* reasoning about experiment with uncertain outcome
- *Possibility space* $\mathcal{X}$ of outcomes
- *Gambles* are real-valued functions of the outcomes;

$$\mathcal{L} = \mathcal{X} \to \mathbb{R}$$

  ($\mathcal{L}$ is assumed to be a linear space)

- *Assessment* is a description of a set of acceptable gambles
- *Natural extension* of an assessment is
  the set of all acceptable gambles implied by
  the agent's rationality criteria (and other assumptions)

Coherence, the classical rationality criteria

Constructive

Positive scaling

If f is acceptable and $\lambda > 0$,
then λf is acceptable.

Addition

If f and g are acceptable,
then $f + g$ is acceptable.

Background

Accepting gain

If f is nonnegative for all outcomes,
then f is acceptable.

Avoiding sure loss

If g is negative for all outcomes,
then g is not acceptable.

Where are the
imprecise probabilities
I came here for!!!

Previsions/Expectations are prices for gambles

- '*Prevision*' and '*Expectation*' are synomyms
- *Prices* are real values interpreted as constant gambles
- Lower prevision $\underline{P}(f)$ is the
  supremum acceptable buying price of f:

  $$\underline{P}(f) = \sup\{\nu \in \mathbb{R} : f - \nu \text{ is acceptable}\}$$

  Upper prevision $\overline{P}(f)$ is the
  infimum acceptable selling price of f
- Conjugacy of coherent lower and upper previsions:

  $$\overline{P}(f) = -\underline{P}(-f)$$

- If $\underline{P}(f) = \overline{P}(f)$, then $P(f) = \underline{P}(f)$ is the *prevision* of f

Probabilities are previsions of indicator gambles

- *Event* A is a subset of $\mathcal{X}$
- *Indicator gamble*

$$1_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A \end{cases}$$

- Lower probability $\underline{P}(A) = \underline{P}(1_A)$

  Upper probability $\overline{P}(A) = \overline{P}(1_A)$
- Conjugacy of coherent lower and upper probabilities ($A^\mathsf{c} = \mathcal{X} \setminus A$):

$$\overline{P}(A) = 1 - \underline{P}(A^\mathsf{c})$$

- If $\underline{P}(A) = \overline{P}(A)$, then $P(A) = \underline{P}(A)$ is the *probability* of A

Natural extension ($\lambda, \mu \geq 0$)	**Agents**
$(\lambda + \mu_{\text{🇧🇪}})$ 🇧🇪 $+(-5\lambda + \mu_{\text{🇯🇵}})$ 🇯🇵	Wiske
$(-4\lambda + \mu_{\text{🇧🇪}})$ 🇧🇪 $(\lambda + \mu_{\text{🇯🇵}})$ 🇯🇵	Yoko Tsuno
$(\lambda_{\mathsf{W}} - 4\lambda_{\mathsf{Y}} + \mu_{\text{🇧🇪}})$ 🇧🇪 $+ (-5\lambda_{\mathsf{W}} + \lambda_{\mathsf{Y}} + \mu_{\text{🇯🇵}})$ 🇯🇵	Irrational pool
$\mu_{\text{🇧🇪}}$ 🇧🇪 $+ \mu_{\text{🇯🇵}}$ 🇯🇵	Rational pool

Wiske's lower probability that Belgium will win

$$\underline{P}(\text{🇧🇪}) = \sup\left\{\nu : \begin{bmatrix}1-\nu\\ -\nu\end{bmatrix} = \begin{bmatrix}\lambda + \mu_{\text{🇧🇪}}\\ -5\lambda + \mu_{\text{🇯🇵}}\end{bmatrix}, \mu \geq 0, \lambda \geq 0\right\}$$

Wiske's lower probability that Belgium will win

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$$= \sup\{5\lambda + \mu_{\text{🇯🇵}} : 1 - 5\lambda + \mu_{\text{🇯🇵}} = \lambda + \mu_{\text{🇧🇪}}, \mu \geq 0, \lambda \geq 0\}$$

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$$= \sup\{5\lambda + \mu_{\text{🇯🇵}} : 1 - 5\lambda + \mu_{\text{🇯🇵}} = \lambda + \mu_{\text{🇧🇪}}, \mu \geq 0, \lambda \geq 0\}$$

$$= \sup\left\{5\lambda + \mu_{\text{🇯🇵}} : \lambda = \frac{1}{6}(1 + \mu_{\text{🇯🇵}} - \mu_{\text{🇧🇪}}), \mu \geq 0, \lambda \geq 0\right\}$$

Wiske's lower probability that Belgium will win

$$\underline{P}(\text{🇧🇪}) = \sup\left\{\nu : \begin{bmatrix} 1-\nu \\ -\nu \end{bmatrix} = \begin{bmatrix} \lambda + \mu_{\text{🇧🇪}} \\ -5\lambda + \mu_{\text{🇯🇵}} \end{bmatrix}, \mu \geq 0, \lambda \geq 0\right\}$$

$$= \sup\{5\lambda + \mu_{\text{🇯🇵}} : 1 - 5\lambda + \mu_{\text{🇯🇵}} = \lambda + \mu_{\text{🇧🇪}}, \mu \geq 0, \lambda \geq 0\}$$

$$= \sup\left\{5\lambda + \mu_{\text{🇯🇵}} : \lambda = \frac{1}{6}(1 + \mu_{\text{🇯🇵}} - \mu_{\text{🇧🇪}}), \mu \geq 0, \lambda \geq 0\right\}$$

$$= \sup\left\{\frac{5}{6} - \frac{1}{6}\mu_{\text{🇯🇵}} - \frac{5}{6}\mu_{\text{🇧🇪}} : \mu \geq 0\right\}$$

$$= \frac{5}{6}$$

$\underline{P}($🇧🇪$), \overline{P}($🇧🇪$)$	$\underline{P}($🇯🇵$), \overline{P}($🇯🇵$)$	**Agents**
$\frac{5}{6}, 1$	$0, \frac{1}{6}$	Wiske
$0, \frac{1}{5}$	$\frac{4}{5}, 1$	Yoko Tsuno
$+\infty, -\infty$	$+\infty, -\infty$	Irrational pool
$0, 1$	$0, 1$	Rational pool

Assessments of lower previsions

- Assume a lower prevision $\underline{P}$ with values assessed for a set of gambles $\mathcal{K}$
- How can we apply the theory we have seen?
- Translate the lower prevision $\underline{P}(f)$ for a gamble $f \in \mathcal{K}$ into a set

$$\{f - \underline{P}(f) + \varepsilon : \varepsilon > 0\}$$

  of acceptable gambles
- The gambles $f - \underline{P}(f)$ are called *marginal gambles*

Expressions for assessments of lower previsions

Avoiding sure loss

$$\sup_{x \in \mathcal{X}} \sum_{k=1}^{n} (f_k(x) - \underline{P}(f_k)) \geq 0 \quad \text{for all } n \geq 0 \text{ and } f_k \in \mathcal{K}$$

Expressions for assessments of lower previsions

Avoiding sure loss

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Coherence

$$\sup_{x\in\mathcal{X}} \left(\sum_{k=1}^{n} (f_k(x) - \underline{P}(f_k)) - m(f_0 - \underline{P}(f_0)) \right) \geq 0$$

for all $n, m \geq 0$ and $f_k \in \mathcal{K}$

Expressions for assessments of lower previsions

Avoiding sure loss

$$\sup_{x\in\mathcal{X}} \sum_{k=1}^{n} \left(f_k(x) - \underline{P}(f_k)\right) \geq 0 \quad \text{for all } n \geq 0 \text{ and } f_k \in \mathcal{K}$$

Coherence

$$\sup_{x\in\mathcal{X}} \left(\sum_{k=1}^{n} \left(f_k(x) - \underline{P}(f_k)\right) - m(f_0 - \underline{P}(f_0)) \right) \geq 0$$

for all $n, m \geq 0$ and $f_k \in \mathcal{K}$

Natural extension

$$\underline{E}(f) = \sup\left\{ \inf_{x\in\mathcal{X}} \left\{ f(x) - \sum_{k=1}^{n} \lambda_k \left(f_k(x) - \underline{P}(f_k)\right) \right\} : n \geq 0, f_k \in \mathcal{K}, \lambda_k > 0 \right\}$$

Does it really have to be so involved?

Lower previsions on linear spaces

If the lower prevision $\underline{P}$ is defined for all gambles in a linear space $\mathcal{L}$, the coherence criteria simplify:

Accepting sure gains

$$\underline{P}(f) \geq \inf f \quad \text{for all } f \in \mathcal{L}$$

Super-linearity

$$\underline{P}(f+g) \geq \underline{P}(f) + \underline{P}(g) \quad \text{for all } f, g \in \mathcal{L}$$

Positive homogeneity

$$\underline{P}(\lambda f) = \lambda \underline{P}(f) \quad \text{for all } f \in \mathcal{L} \text{ and } \lambda > 0$$

Upper previsions on linear spaces

If the upper prevision $\overline{P}$ is defined for all gambles in a linear space $\mathcal{L}$, the coherence criteria simplify:

Accepting sure gains

$$\overline{P}(f) \leq \sup f \quad \text{for all } f \in \mathcal{L}$$

Sub-linearity

$$\overline{P}(f+g) \leq \overline{P}(f) + \overline{P}(g) \quad \text{for all } f, g \in \mathcal{L}$$

Positive homogeneity

$$\overline{P}(\lambda f) = \lambda \overline{P}(f) \quad \text{for all } f \in \mathcal{L} \text{ and } \lambda > 0$$

Coherent lower & upper previsions

For a coherent lower prevision $\underline{P}$ and its conjugate upper prevision $\overline{P}$ many useful properties can be derived; we present a few:

Upper dominates lower $\overline{P}(f) \geq \underline{P}(f)$ for all $f \in \mathcal{L}$

Constants $\overline{\underline{P}}(\mu) = \mu$ for all $\mu \in \mathbb{R}$

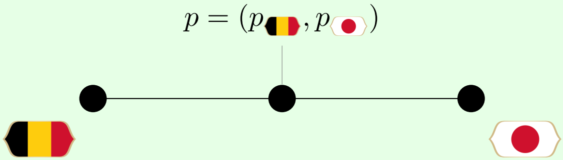
Constant additivity $\overline{\underline{P}}(f + \mu) = \overline{\underline{P}}(f) + \mu$ for all $f \in \mathcal{L}$ and $\mu \in \mathbb{R}$

Gamble dominance
if $f \geq g + \mu$ then $\overline{\underline{P}}(f) \geq \overline{\underline{P}}(g) + \mu$ for all $f, g \in \mathcal{L}$ and $\mu \in \mathbb{R}$

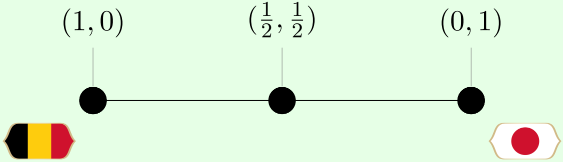
Mixed sub/super-additivity
$$\underline{P}(f + g) \leq \underline{P}f + \overline{P}(g) \leq \overline{P}(f + g) \quad \text{for all } f, g \in \mathcal{L}$$

I heard that imprecise probabilities are just *sets of probabilities*?

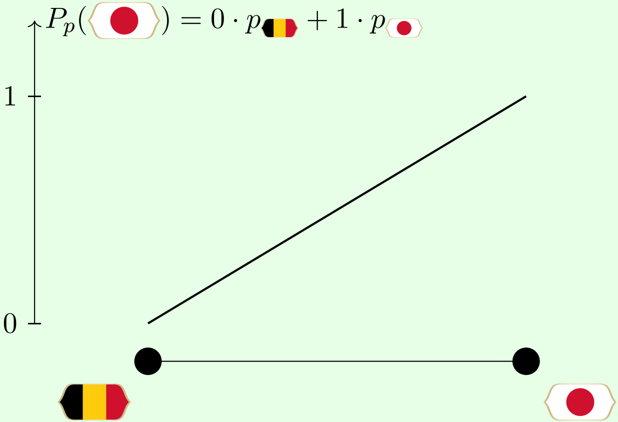
The probability simplex



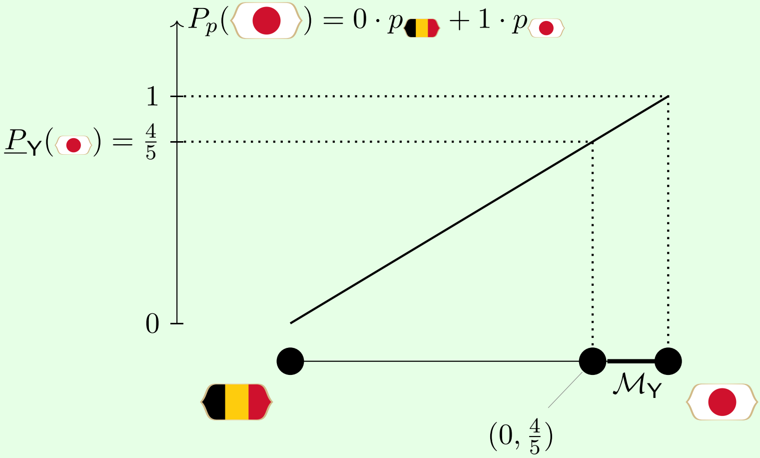
The probability simplex



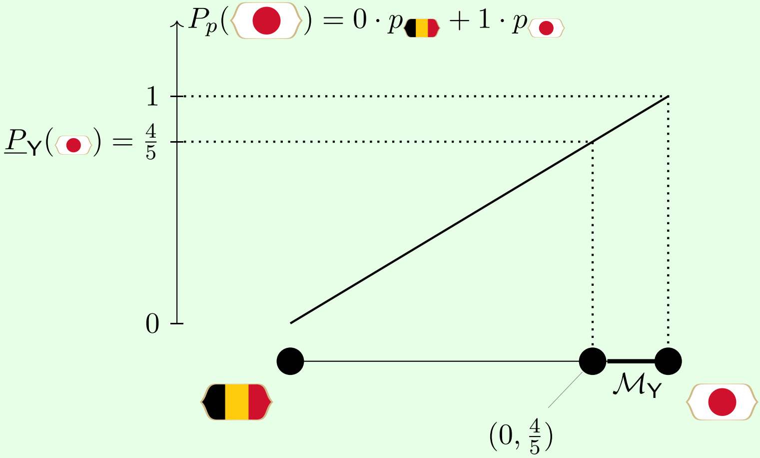
The probability simplex



The probability simplex



The probability simplex



CREDAL SET

From lower previsions to credal sets

- The prevision of a gamble
  is a linear function over the probability simplex

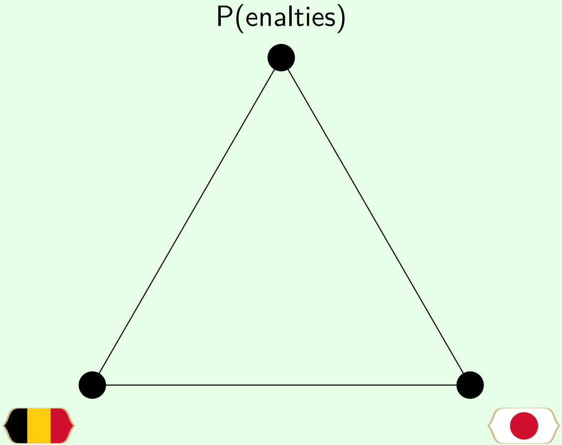
- The lower prevision of a gamble
  can be seen as bounding the prevision of that gamble
  so constraining the possible probability mass functions

- A lower prevision corresponds to a set of constraints,
  defining a credal set (closed convex set)

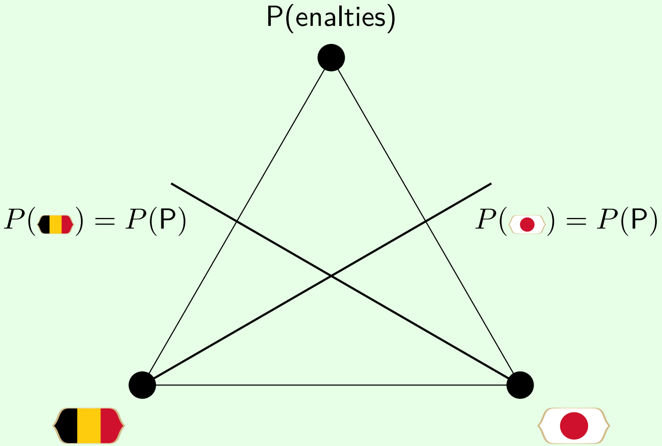
$$\mathcal{M} = \{p : P_p(f) \geq \underline{P}(f) \text{ for all } f \in \mathcal{K}\}$$

- All this generalizes to infinite $\mathcal{X}$

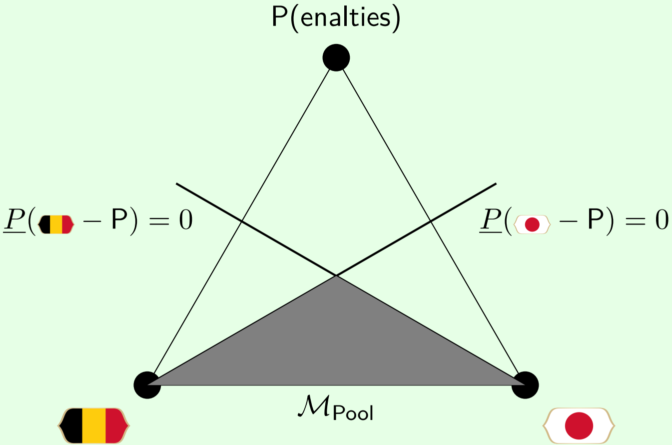
A larger probability simplex



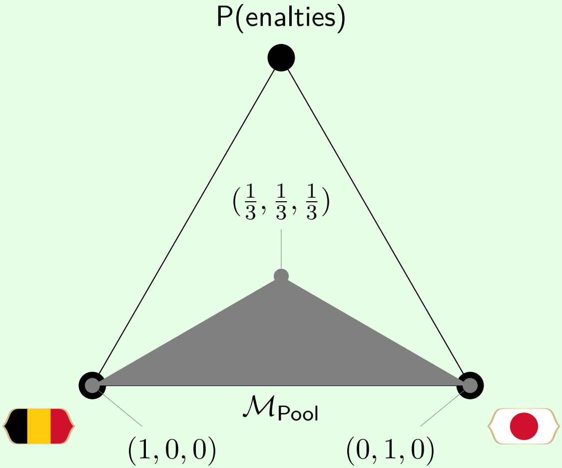
A larger probability simplex



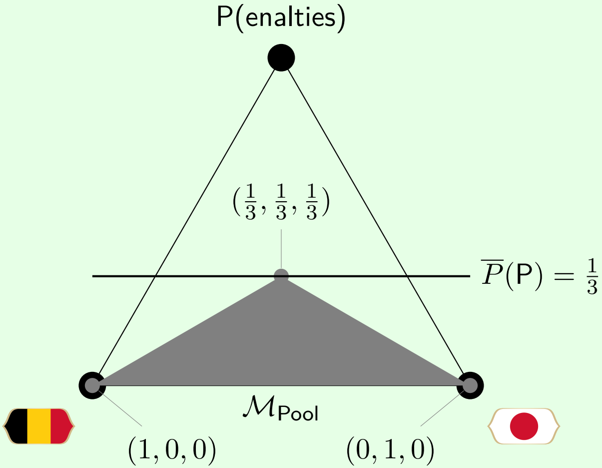
A larger probability simplex



A larger probability simplex



A larger probability simplex



From credal sets to lower previsions

- A *credal set* is a closed convex set of probability mass functions
  (or more generally, previsions)

- A credal set is determined completely by its set of *extreme points* $\mathcal{M}^*$

- A nonempty credal set is equivalent to a coherent lower prevision

Lower envelope theorem

$$
\begin{aligned}
\underline{P}(f) &= \min\{P_p(f) : p \in \mathcal{M}\} \\
&= \min\{P_p(f) : p \in \mathcal{M}^*\}
\end{aligned}
$$

Where are the conditional models?
We need them to learn!

Basics of conditioning and updating

- Conditioning acceptable gambles is done by restriction to the subspace of gambles that are zero outside the conditioning event
- Conditioning lower previsions is a form of natural extension

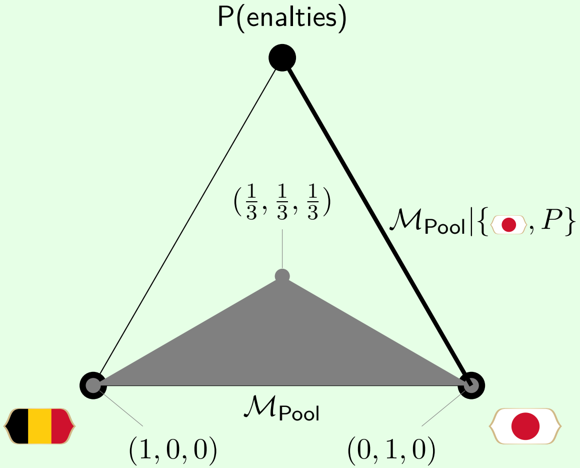
$$\underline{E}(f \mid A) = \begin{cases} \inf_{x \in A} f(x) & \text{if } \underline{P}(A) = 0, \\ \max\{\mu \in \mathbb{R} : \underline{P}(1_A(f - \mu))\} & \text{if } \underline{P}(A) > 0 \end{cases}$$

- Conditional credal set =
  credal set of conditional probability mass functions

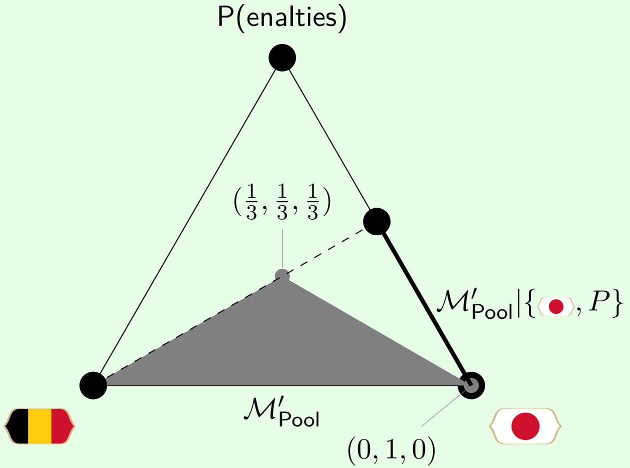
$$\mathcal{M}|A = \begin{cases} \text{whole simplex} & \text{if } \exists p \in \mathcal{M} : P_p(A) = 0, \\ \{p(\cdot \mid A) : p \in \mathcal{M}\} & \text{if } \forall p \in \mathcal{M} : P_p(A) > 0 \end{cases}$$

- Natural extension often gives vacuous conditionals;
  regular extension is a less imprecise updating rule:
  it removes those p such that $P_p(A) = 0$ from $\mathcal{M}$

Conditioning using natural extension



Conditioning using regular extension

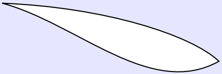


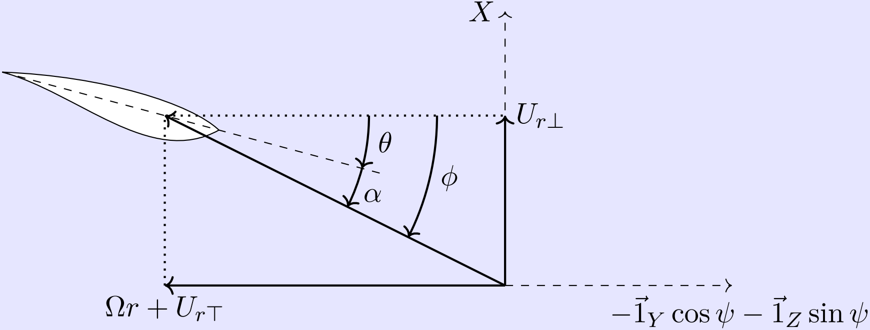
What about imprecise probabilities on continuous spaces?

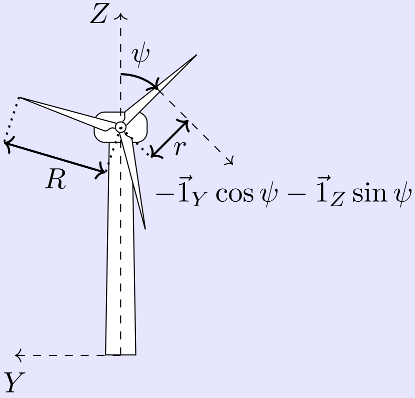
Too few remarks about imprecise probabilities on continuous spaces

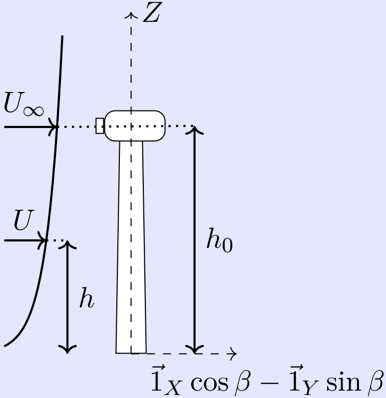
- Mostly defined using credal sets
  of parametric distributions
  where the parameters vary in a set
- Also common:
  defined using probability mass assignments
  to subsets of the space
- Examples:
  - Imprecise Dirichlet model
  - P-boxes
  - lower density functions
- Calculating lower and upper previsions (natural extension)
  can easily become difficult optimization problems

Nothing in this lecture had anything to do with aerospace engineering!




X
$U_{r\perp}$
θ
α
ϕ
$\Omega r + U_{r\top}$
$-\vec{1}_Y \cos\psi - \vec{1}_Z \sin\psi$


Z
ψ
r
R
$-\vec{1}_Y \cos\psi - \vec{1}_Z \sin\psi$
Y


Z
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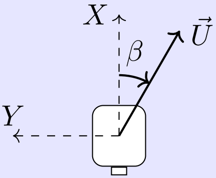
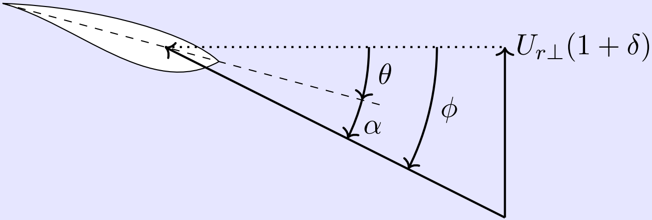
U
h
h_0
$\vec{1}_X \cos\beta - \vec{1}_Y \sin\beta$


X
$\vec{U}$
β
Y

Relationship between turbulence and angle of attack



$$\alpha_\delta = \alpha - \alpha_0 = \operatorname{atan}\left((1+\delta)\tan\phi_0\right) - \phi_0$$

$$\delta = \frac{\tan(\phi_0 + \alpha_\delta) - \tan\phi_0}{\tan\phi_0}$$

Is this all you have?