Modeling Uncertainty
using
Accept & Reject Statements

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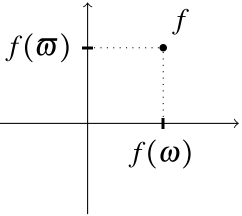
The setup

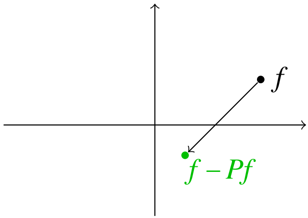
- Experiment with outcomes in some possibility space Ω.
- Agent uncertain about the experiment's outcome.
- Linear space $\mathcal{L}$ of real-valued gambles on Ω.



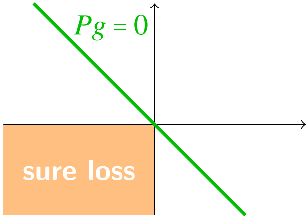
- Agent expresses uncertainty by making statements about gambles, forming an assessment.
- Agent wishes to rationally deduce inferences and draw conclusions from this assessment.

The work we build on

- De Finetti: previsions P.



$\Rightarrow$



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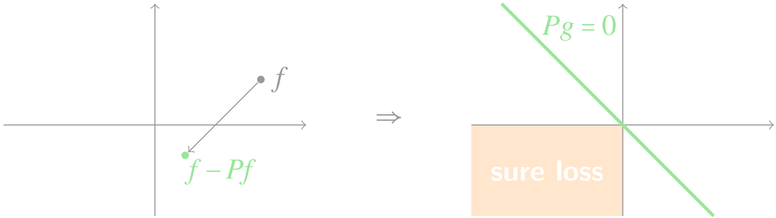
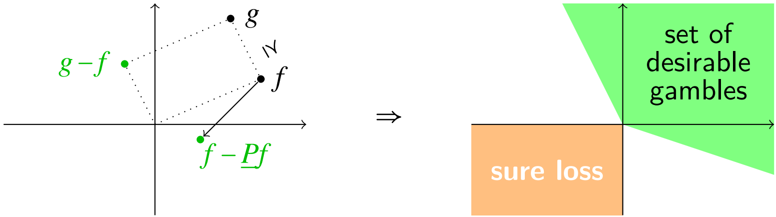


- Williams, Seidenfeld et al., Walley:
  - lower previsions $\underline{P}$,
  - sets of acceptable/favorable/desirable gambles,
  - partial preference orders $\geq$.



Accepting & Rejecting Gambles

Accepting a gamble f implies a commitment to engage in the following transaction:

(i) the experiment's outcome $\omega \in \Omega$ is determined,

(ii) the agent gets the—possibly negative—payoff $f(\omega)$.

Rejecting a gamble: the agent considers accepting it unreasonable.

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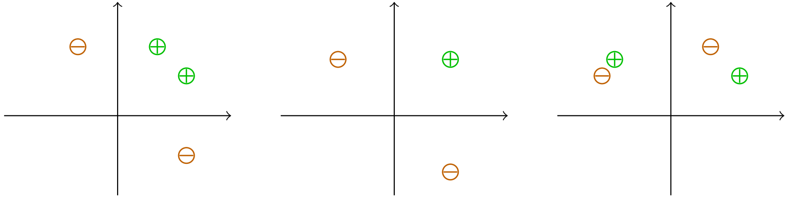
Assessment A pair $\mathcal{A} \coloneqq \langle \mathcal{A}_{\geq}; \mathcal{A}_{\prec} \rangle$ of sets of accepted and rejected gambles.



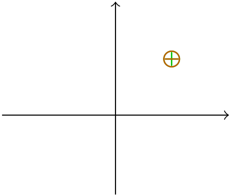
Gamble Categorization

Accepted $\mathcal{A}_{\geq}$.

Rejected $\mathcal{A}_{\prec}$.

Unresolved Neither accepted nor rejected; $\mathcal{A}_{\smile} \coloneqq \mathcal{L} \setminus (\mathcal{A}_{\geq} \cup \mathcal{A}_{\prec})$.

Confusing Both accepted and rejected; $\mathcal{A}_{\geq,\prec} \coloneqq \mathcal{A}_{\geq} \cap \mathcal{A}_{\prec}$.



Gamble Categorization

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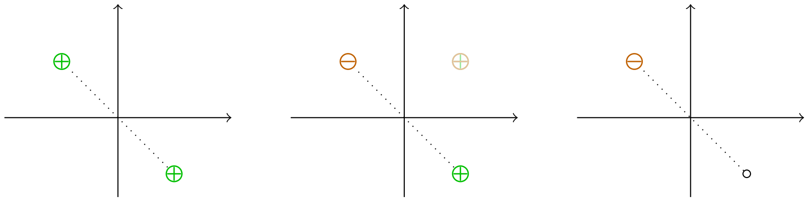

Indifferent Both it and its negation accepted; $\mathcal{A}_{\simeq} \coloneqq \mathcal{A}_{\succeq} \cap -\mathcal{A}_{\succeq}$.

Favorable Accepted with a rejected negation; $\mathcal{A}_{\succ} \coloneqq \mathcal{A}_{\succeq} \cap -\mathcal{A}_{\prec}$.

Indeterminate Both it and its negation not acceptable; $\mathcal{A}_{\parallel} \coloneqq (\mathcal{A}_{\succeq} \cup -\mathcal{A}_{\succeq})^c$.

Axiom: **No Confusion**

Because of the interpretation attached to acceptance and rejection statements, we consider confusion irrational.

So we require assessments $\mathcal{A}$ to not contain confusion:

$$\mathcal{A}_{\geq,\prec} = \mathcal{A}_{\geq} \cap \mathcal{A}_{\prec} = \emptyset$$

Axiom *template*: **Background Model**

Problem domain specific set of acceptable gambles $\mathcal{S}_{\geq}$ and set of rejected gambles $\mathcal{S}_{<}$. To be combined with the agent's own assessment.

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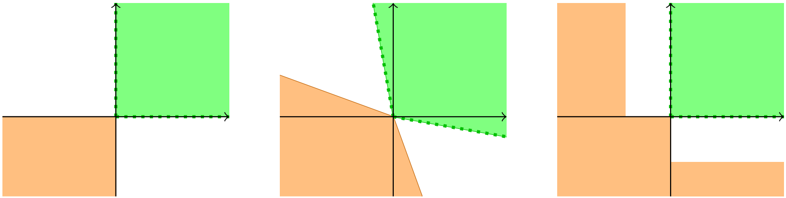

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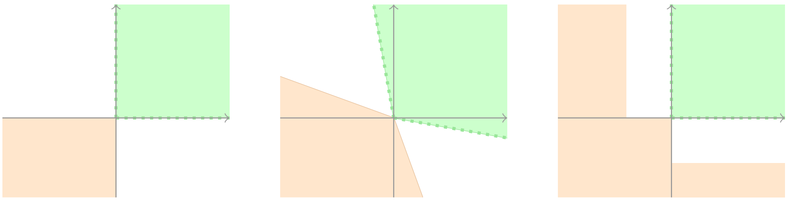

For convenience, assume Indifference to Status Quo: $0 \in \mathcal{S}_{\geq}$.

Deductive extension

The nature of the gamble payoffs (utility considerations) determines a deductive extension rule for acceptable gambles: given a set of acceptable gambles, which other gambles should be acceptable to the agent.

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1. Positive linear combinations (assumption of linear precise utility):
   - sums of accepted gambles are acceptable ($\mathcal{A} + \mathcal{A} \subseteq \mathcal{A}$).
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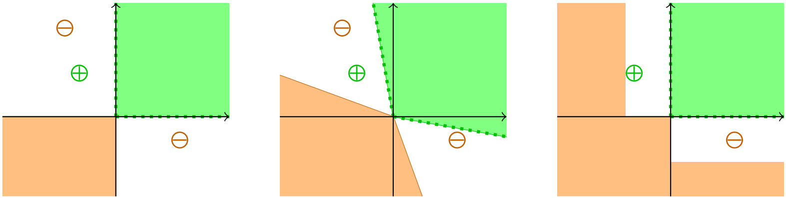
The positive linear hull operator posi combines both operations;



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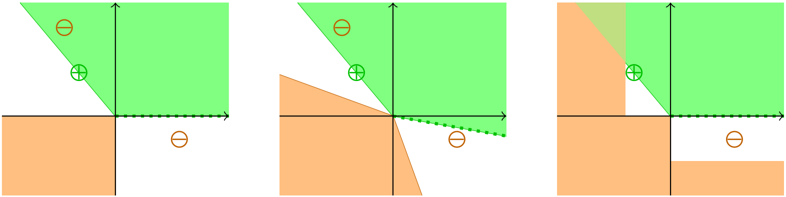
The positive linear hull operator posi combines both operations; it generates convex cones.



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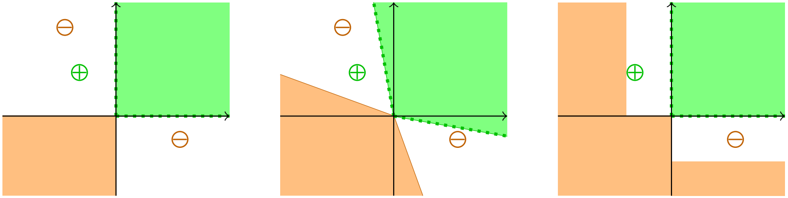

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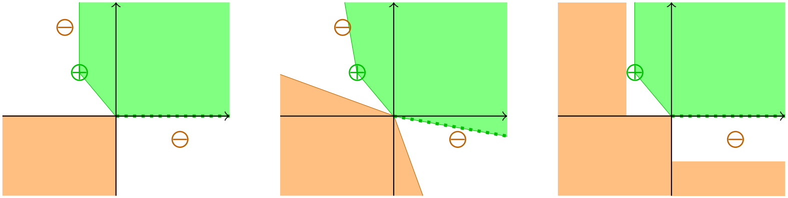

Axiom *template*: **Deductive Closure**

An assessment $\mathcal{A}$ can be deductively extended to a deductively closed assessment $\mathcal{D}$;

1. $\mathcal{D} \coloneqq \langle \operatorname{posi} \mathcal{A}_{\geq}; \mathcal{A}_{\prec} \rangle$,
2. $\mathcal{D} \coloneqq \langle \operatorname{co} \mathcal{A}_{\geq}; \mathcal{A}_{\prec} \rangle$.

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The assumptions underlying the choice of a deductive extension rule lead us to exclusively use deductively closed assessments $\mathcal{D}$ for inference and decision purposes:

1. $\operatorname{posi} \mathcal{D}_{\geq} = \mathcal{D}_{\geq}$
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Gambles in limbo & reckoning extension

Deductive Closure interacts with No Confusion:

- Consider a deductively closed assessment $\mathcal{D}$.
- Additionally consider some unresolved gamble f acceptable.
- Apply deductive extension to $\langle \mathcal{D}_{\succeq} \cup \{f\}; \mathcal{D}_{\prec} \rangle$.
- For some f, this would lead to an increase in confusion.
- These have the same effect as gambles in $\mathcal{D}_{\prec}$,
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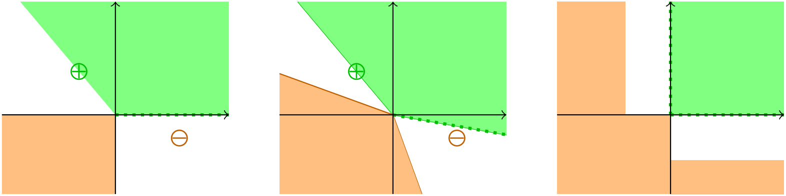

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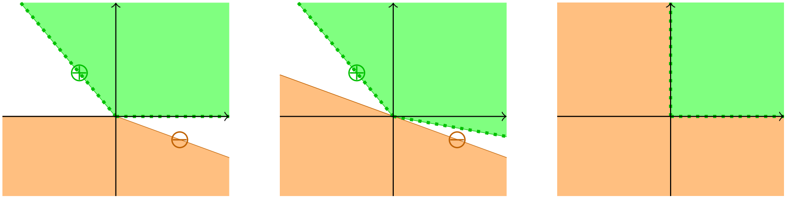

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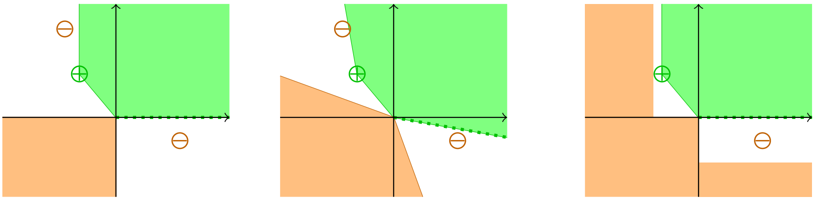

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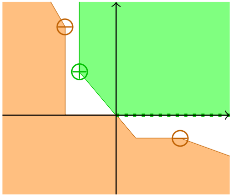
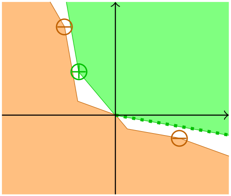
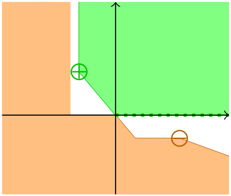
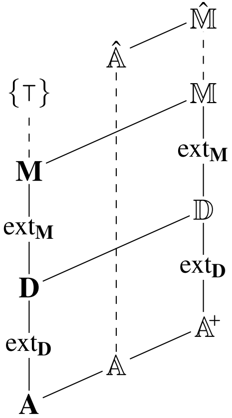




Axiom: **No Limbo**

We consider accepting gambles in limbo unreasonable and therefore further restrict attention to models $\mathcal{M}$ for inference and decision purposes:

1. $\overline{\mathcal{M}_{\prec}} - \mathcal{M}_{\succeq} \subseteq \mathcal{M}_{\prec}$
2. $\bigcup_{\mu>0} (\mu+1)\mathcal{M}_{\prec} - \mu\mathcal{M}_{\succeq} \subseteq \mathcal{M}_{\prec}$

Order-theoretic considerations



Main characterization result (posi)

An assessment $\mathcal{M}$ is a model that satisfies
No Confusion and Indifference to Status Quo iff

(i) $0 \in \mathcal{M}_{\geq}$,

(ii) $0 \notin \mathcal{M}_{\prec}$,

(iii) $\operatorname{posi} \mathcal{M}_{\geq} = \mathcal{M}_{\geq}$,

(iv) $\overline{\mathcal{M}_{\prec}} - \mathcal{M}_{\geq} \subseteq \mathcal{M}_{\prec}$.

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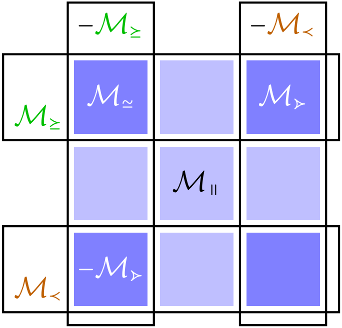
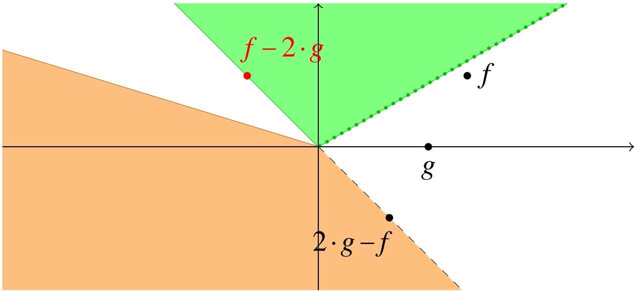
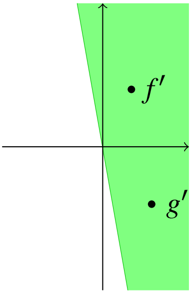
These partition gamble space as follows:



Gamble relations (posi)

- f is accepted in exchange for h: $f \geq h \Leftrightarrow f - h \in \mathcal{M}_{\geq}$.





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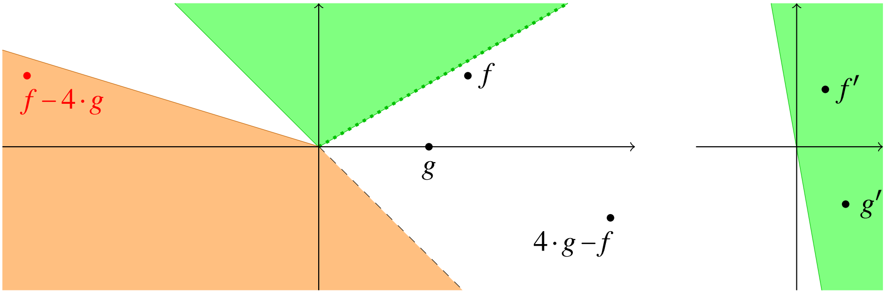

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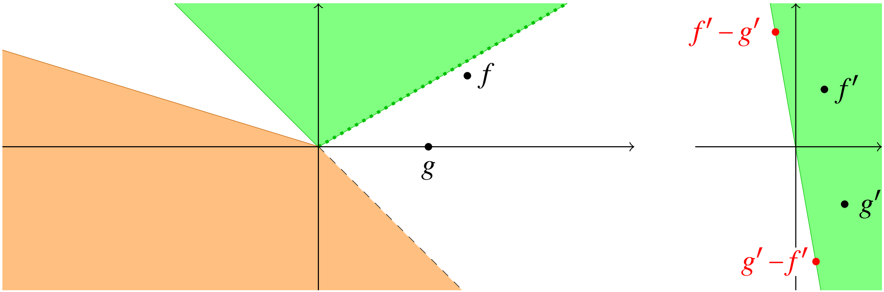

- indifference between f and h: $f \simeq h \Leftrightarrow f \succeq h \wedge h \succeq f \Leftrightarrow f - h \in \mathcal{M}_{\simeq}$.

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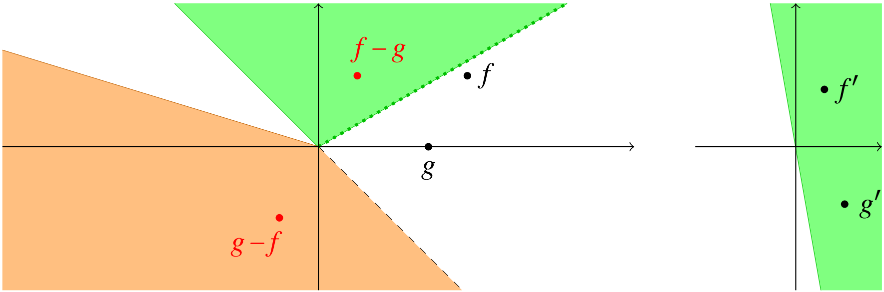

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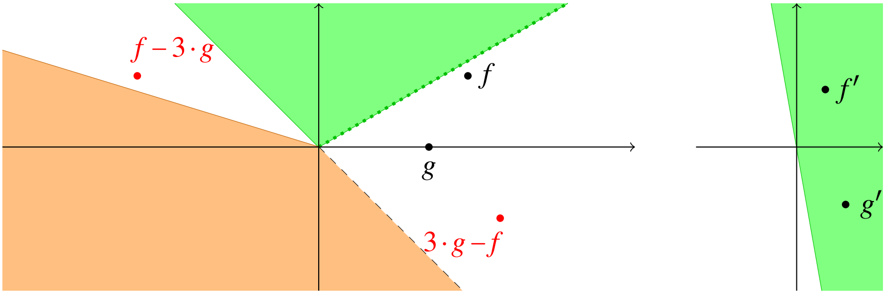

- indifference between f and h: $f \simeq h \Leftrightarrow f \geq h \wedge h \geq f \Leftrightarrow f - h \in \mathcal{M}_{\simeq}$.
- f is preferred over h: $f \succ h \Leftrightarrow f \geq h \wedge h \prec f \Leftrightarrow f - h \in \mathcal{M}_{\succ}$.
- f and h are uncomparable: $f \parallel h \Leftrightarrow f - h \in \mathcal{M}_{\parallel}$.

Characterization result for gamble relations (posi)

Gamble relations $\succeq$ and $\prec$ are equivalent to a model that satisfies No Confusion and Indifference to Status Quo iff

(i) Accept Reflexivity: $f \succeq f$,

(ii) Reject Irreflexivity: $f \not\prec f$,

(iii) Accept Transitivity: $f \succeq g \wedge g \succeq h \Rightarrow f \succeq h$.

(iv) Mixed Transitivity: $f \prec g \wedge h \succeq g \Rightarrow f \prec h$,

(v) Mixture independence: $f \succeq g \Leftrightarrow \mu \cdot f + (1-\mu) \cdot h \succeq \mu \cdot g + (1-\mu) \cdot h$.

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Gamble relations $\geq$ and $\prec$ are equivalent to a model that satisfies No Confusion and Indifference to Status Quo iff

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(iii) Accept Transitivity: $f \geq g \wedge g \geq h \Rightarrow f \geq h$.

(iv) Mixed Transitivity: $f \prec g \wedge h \geq g \Rightarrow f \prec h$,

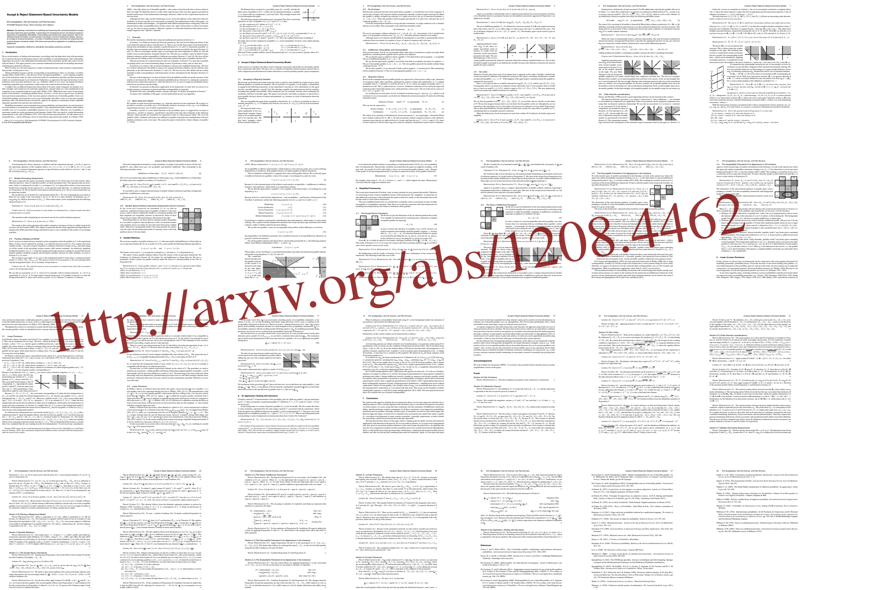
(v) Mixture independence: $f \geq g \Leftrightarrow \mu \cdot f + (1-\mu) \cdot h \geq \mu \cdot g + (1-\mu) \cdot h$.

- Acceptability $\geq$ is a non-strict pre-order (a vector ordering).
- Indifference $\simeq$ is an equivalence relation.
- Preference $\succ$ is a strict partial order.

Conclusions

- Our framework further generalizes existing generalizations of probability theory.
- The generalization is flexible on input (assessment/elicitation) and output (inference/decisions) side.
- It allows for interesting model types: choose appropriate background models and deductive closure axioms.
- It elegantly combines distinct strict and non-strict preference orders,

Want to know more: read the full paper!



http://arxiv.org/abs/1208.4462