A *propositional* CONEstrip algorithm

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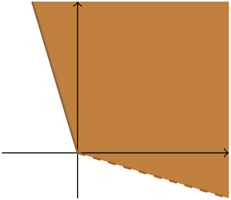
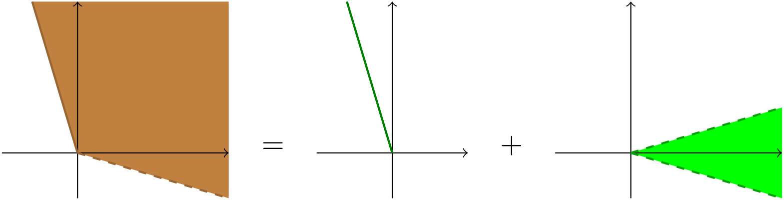
Which cones? *General* cones of gambles.



$\underline{\mathcal{R}}$

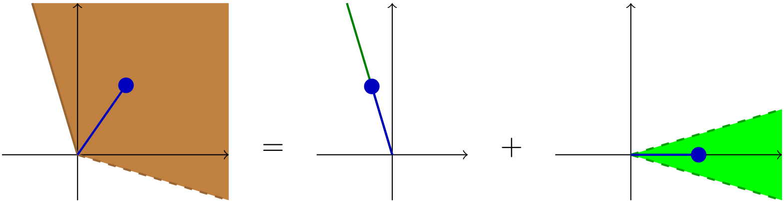
Decomposing a *general* cone into *open* cones



$$\underline{\mathcal{R}} \quad = \quad \lambda \quad \hat{\mathcal{D}} \qquad\qquad + \quad (1-\lambda) \quad \check{\mathcal{D}}$$

$$0 \leq \lambda \leq 1$$

Our goal: Answering '*Does this gamble lie in that cone?*'



$$\underline{\mathcal{R}} = \lambda \quad \hat{\mathcal{D}} \qquad + \quad (1-\lambda) \quad \check{\mathcal{D}}$$

$$f \stackrel{?}{=} \lambda \sum_{g \in \hat{\mathcal{D}}} \hat{\nu}_g g \qquad + \quad (1-\lambda) \sum_{g \in \check{\mathcal{D}}} \check{\nu}_g g$$

$0 \leq \lambda \leq 1$ and $\nu > 0$

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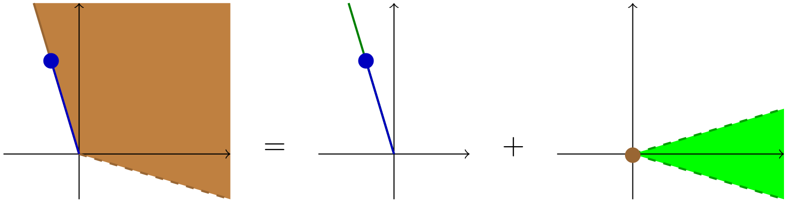

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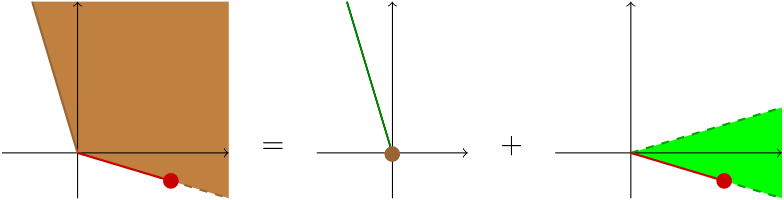

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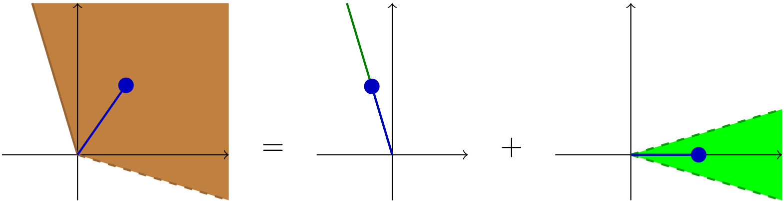

$$\underline{\mathcal{R}} = \lambda \; \hat{\mathcal{D}} + (1-\lambda) \; \check{\mathcal{D}}$$

$$f \stackrel{?}{=} \lambda \sum_{g \in \hat{\mathcal{D}}} \hat{\nu}_g g + (1-\lambda) \sum_{g \in \check{\mathcal{D}}} \check{\nu}_g g$$

$0 \leq \lambda \leq 1$ and $\nu > 0$

Why?

Finitary (imprecise) probabilistic deductive inference reduces to such feasibility problems or optimization variants thereof.

Problem: The cones are *really big*

How big?

> *With dimension exponential in the number of events involved in the assessment represented by the cone.*

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So for n events...

> 2^n *dimensions*

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How many?

Polynomial in n

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How big?

With dimension exponential in the number of events involved in the assessment represented by the cone.

So for n events...

2^n *dimensions*

How many?

Polynomial in n

Any downsides or trade-offs?

Computationally 'hard' to find the right dimensions.

The general, nonlinear problem with strict inequalities

find $\lambda_{\mathcal{D}} \in [0,1]$ and $\nu_{\mathcal{D}} \in (\mathbb{R}_{>0})^{\mathcal{D}}$ for all $\mathcal{D}$ in $\mathcal{R}_0$

s.t. $\sum_{\mathcal{D}\in\mathcal{R}_0} \lambda_{\mathcal{D}} = 1$ and $\sum_{\mathcal{D}\in\mathcal{R}_0} \lambda_{\mathcal{D}} \sum_{g\in\mathcal{D}} \nu_{\mathcal{D},g}\, g \gtreqless f$

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$h \lesseqgtr f$ means $\begin{array}{l} h(\omega) \leq f(\omega) \text{ for } \omega \text{ in } \Omega_\Gamma \\ h(\omega) \geq f(\omega) \text{ for } \omega \text{ in } \Omega_\Delta \end{array}$ with $\Omega_\Gamma \cup \Omega_\Delta = \Omega$

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s.t. $\quad \sum_{\mathcal{D}\in\mathcal{R}_0} \lambda_{\mathcal{D}} \geq 1 \quad$ and $\quad \sum_{\mathcal{D}\in\mathcal{R}_0} \lambda_{\mathcal{D}} \sum_{g\in\mathcal{D}} \nu_{\mathcal{D},g}\, g \overset{\times}{\leq} f$

$$h \overset{\times}{\leq} f \quad \text{means} \quad \begin{array}{l} h(\omega) \leq f(\omega) \text{ for } \omega \text{ in } \Omega_\Gamma \\ h(\omega) \geq f(\omega) \text{ for } \omega \text{ in } \Omega_\Delta \end{array} \quad \text{with } \Omega_\Gamma \cup \Omega_\Delta = \Omega$$

The general, nonlinear problem

find $\lambda_{\mathcal{D}} \in [0,1]$ and $\tau_{\mathcal{D}} \in (\mathbb{R}_{\geq 1})^{\mathcal{D}}$ for all $\mathcal{D}$ in $\mathcal{R}_0$ and $\sigma \in \mathbb{R}_{\geq 1}$

s.t. $\sum_{\mathcal{D}\in\mathcal{R}_0} \lambda_{\mathcal{D}} \geq 1$ and $\sum_{\mathcal{D}\in\mathcal{R}_0} \lambda_{\mathcal{D}} \sum_{g\in\mathcal{D}} \tau_{\mathcal{D},g}\, g \lesseqqgtr \sigma f$

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The general problem, with isolated nonlinearities

find $\lambda_{\mathcal{D}} \in [0,1]$ and $\mu_{\mathcal{D}} \in (\mathbb{R}_{\geq 0})^{\mathcal{D}}$ for all $\mathcal{D}$ in $\mathcal{R}_0$ and $\sigma \in \mathbb{R}_{\geq 1}$

s.t. $\sum_{\mathcal{D}\in\mathcal{R}_0} \lambda_{\mathcal{D}} \geq 1$ and $\sum_{\mathcal{D}\in\mathcal{R}_0}\sum_{g\in\mathcal{D}}\mu_{\mathcal{D},g}\, g \lesseqgtr \sigma f$

$\lambda_{\mathcal{D}} \leq \mu_{\mathcal{D}} \leq \lambda_{\mathcal{D}}\mu_{\mathcal{D}}$ for all $\mathcal{D}$ in $\mathcal{R}_0$

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find $\lambda_{\mathcal{D}} \in [0,1]$ and $\mu_{\mathcal{D}} \in (\mathbb{R}_{\geq 0})^{\mathcal{D}}$ for all $\mathcal{D}$ in $\mathcal{R}_0$ and $\sigma \in \mathbb{R}_{\geq 1}$

s.t. $\sum_{\mathcal{D}\in\mathcal{R}_0} \lambda_{\mathcal{D}} \geq 1$ and $\sum_{\mathcal{D}\in\mathcal{R}_0}\sum_{g\in\mathcal{D}}\mu_{\mathcal{D},g}\, g \lesseqgtr \sigma f$

$\lambda_{\mathcal{D}} \leq \mu_{\mathcal{D}} \leq \lambda_{\mathcal{D}}\mu_{\mathcal{D}}$ for all $\mathcal{D}$ in $\mathcal{R}_0$ (forces $\lambda_{\mathcal{D}} \in \{0,1\}$)

$h \lesseqgtr f$ means $h(\omega) \leq f(\omega)$ for ω in Ω_Γ, $h(\omega) \geq f(\omega)$ for ω in Ω_Δ with $\Omega_\Gamma \cup \Omega_\Delta = \Omega$

The *plain* CONEstrip algorithm

Init. Set $i := 0$.

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Iter. Does the linear programming problem below have a solution $(\bar{\lambda}, \bar{\mu}, \bar{\sigma})$?

$$
\begin{aligned}
\text{max.} \quad & \textstyle\sum_{\mathcal{D}\in\mathcal{R}_i} \lambda_{\mathcal{D}} \\
\text{s.t.} \quad & \lambda_{\mathcal{D}} \in [0,1] \quad \text{and} \quad \mu_{\mathcal{D}} \in (\mathbb{R}_{\geq 0})^{\mathcal{D}} \quad \text{for all } \mathcal{D} \text{ in } \mathcal{R}_i \quad \text{and} \quad \sigma \in \mathbb{R}_{\geq 1} \\
& \textstyle\sum_{\mathcal{D}\in\mathcal{R}_i} \lambda_{\mathcal{D}} \geq 1 \quad \text{and} \quad \sum_{\mathcal{D}\in\mathcal{R}_i}\sum_{g\in\mathcal{D}} \mu_{\mathcal{D},g}\, g \lesseqgtr \sigma f \\
& \lambda_{\mathcal{D}} \leq \mu_{\mathcal{D}} \text{ for all } \mathcal{D} \text{ in } \mathcal{R}_i.
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s.t. $\lambda_{\mathcal{D}} \in [0,1]$ and $\mu_{\mathcal{D}} \in (\mathbb{R}_{\geq 0})^{\mathcal{D}}$ for all $\mathcal{D}$ in $\mathcal{R}_i$ and $\sigma \in \mathbb{R}_{\geq 1}$

$\sum_{\mathcal{D} \in \mathcal{R}_i} \lambda_{\mathcal{D}} \geq 1$ and $\sum_{\mathcal{D} \in \mathcal{R}_i} \sum_{g \in \mathcal{D}} \mu_{\mathcal{D},g} g \lesseqgtr \sigma f$

$\lambda_{\mathcal{D}} \leq \mu_{\mathcal{D}}$ for all $\mathcal{D}$ in $\mathcal{R}_i$.

No. $f \notin \underline{\mathcal{R}_0}$. ***Stop***.

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$\sum_{\mathcal{D}\in\mathcal{R}_i} \lambda_{\mathcal{D}} \geq 1$ and $\sum_{\mathcal{D}\in\mathcal{R}_i}\sum_{g\in\mathcal{D}}\mu_{\mathcal{D},g}g \gneqq \sigma f$

$\lambda_{\mathcal{D}} \leq \mu_{\mathcal{D}}$ for all $\mathcal{D}$ in $\mathcal{R}_i$.

No. $f \notin \underline{\mathcal{R}_0}$. ***Stop.***

Yes. Let $\mathcal{Q} := \{\mathcal{D} \in \mathcal{R}_i \colon \bar{\lambda}_{\mathcal{D}} = 0\}$ and set $\mathcal{R}_{i+1} := \mathcal{R}_i \setminus \mathcal{Q}$.

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Is $\{\mathcal{D} \in \mathcal{Q} \colon \bar{\mu}_{\mathcal{D}} = 0\} = \mathcal{Q}$?

The *plain* CONEstrip algorithm

Init. Set $i \coloneqq 0$.

Iter. Does the linear programming problem below have a solution $(\bar{\lambda}, \bar{\mu}, \bar{\sigma})$?

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& \lambda_{\mathcal{D}} \leq \mu_{\mathcal{D}} \text{ for all } \mathcal{D} \text{ in } \mathcal{R}_i.
\end{aligned}$$

No. $f \notin \underline{\mathcal{R}_0}$. ***Stop***.

Yes. Let $\mathcal{Q} \coloneqq \{\mathcal{D} \in \mathcal{R}_i \colon \bar{\lambda}_{\mathcal{D}} = 0\}$ and set $\mathcal{R}_{i+1} \coloneqq \mathcal{R}_i \setminus \mathcal{Q}$.

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Yes. Set $t \coloneqq i+1$; $f \in \underline{\mathcal{R}_t} \subseteq \underline{\mathcal{R}_0}$. ***Stop***.

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$$\lambda_{\mathcal{D}} \leq \mu_{\mathcal{D}} \text{ for all } \mathcal{D} \text{ in } \mathcal{R}_i.$$

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Yes. Let $\mathcal{Q} := \{\mathcal{D} \in \mathcal{R}_i \colon \bar{\lambda}_{\mathcal{D}} = 0\}$ and set $\mathcal{R}_{i+1} := \mathcal{R}_i \setminus \mathcal{Q}$.

Is $\{\mathcal{D} \in \mathcal{Q} \colon \bar{\mu}_{\mathcal{D}} = 0\} = \mathcal{Q}$?

- ***Yes***. Set $t := i + 1$; $f \in \underline{\mathcal{R}_t} \subseteq \underline{\mathcal{R}_0}$. ***Stop***.
- ***No***. Increase i's value by 1. ***Reiterate***.

Structured gambles

Assume

$$g = \sum_{\phi \in \Phi} g_\phi \phi \quad \text{for all } g \text{ in } \{f\} \cup \bigcup \mathcal{R}_0,$$

with Φ a finite set of basic functions.

Structured gambles generate structured problems

Assume

$$g = \sum_{\phi \in \Phi} g_\phi \phi \quad \text{for all } g \text{ in } \{f\} \cup \bigcup \mathcal{R}_0,$$

with Φ a finite set of basic functions.

Then

$$\sum_{\mathcal{D} \in \mathcal{R}_i} \sum_{g \in \mathcal{D}} \mu_{\mathcal{D},g} g \gneqq \sigma f \quad \text{iff} \quad \sum_{\phi \in \Phi} \kappa_\phi \phi \gneqq 0$$

when

$$\kappa_\phi := \Big(\sum_{\mathcal{D} \in \mathcal{R}_i} \sum_{g \in \mathcal{D}} \mu_{\mathcal{D},g} g_\phi \Big) - \sigma f_\phi \quad \text{for all } \phi \text{ in } \Phi.$$

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Then

$$\sum_{\mathcal{D} \in \mathcal{R}_i} \sum_{g \in \mathcal{D}} \mu_{\mathcal{D},g} g \gtreqqless \sigma f \quad \text{iff} \quad \sum_{\phi \in \Phi} \kappa_\phi \phi \gtreqqless 0$$

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$$\kappa_\phi := \Big(\sum_{\mathcal{D} \in \mathcal{R}_i} \sum_{g \in \mathcal{D}} \mu_{\mathcal{D},g} g_\phi \Big) - \sigma f_\phi \quad \text{for all } \phi \text{ in } \Phi.$$

Why?

The now explicit structure can be exploited to solve the problem more efficiently even though we have added constraints.

Row generation: *removing constraints*

Remove

$$\sum_{\phi \in \Phi} \kappa_\phi \phi \gtreqless 0$$

(this relaxes the problem)

Row generation: *adding constraints* back

Remove

$$\sum_{\phi \in \Phi} \kappa_\phi \phi \gtreqless 0$$

(this relaxes the problem)

Iteratively, add back

$$\sum_{\phi \in \Phi} \kappa_\phi \phi(\omega) \gtreqless 0$$

for some wisely chosen ω in Ω:

Row generation: *adding constraints* back intelligently

Remove

$$\sum_{\phi \in \Phi} \kappa_\phi \phi \gtreqless 0$$

(this relaxes the problem)

Iteratively, add back

$$\sum_{\phi \in \Phi} \kappa_\phi \phi(\omega) \gtreqless 0$$

for some wisely chosen ω in Ω:

- Each iteration, a solution $\bar{\kappa}$ is obtained;
  choose the next ω such that $\bar{\kappa}$ violates the corresponding constraint.

Row generation: *adding constraints* back intelligently

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$$\sum_{\phi \in \Phi} \kappa_\phi \phi \lesseqgtr 0$$

(this relaxes the problem)

Iteratively, add back

$$\sum_{\phi \in \Phi} \kappa_\phi \phi(\omega) \lesseqgtr 0$$

for some wisely chosen ω in Ω:

- Each iteration, a solution $\bar{\kappa}$ is obtained;
  choose the next ω such that $\bar{\kappa}$ violates the corresponding constraint.
- A 'deep cut' is generated by solving the problem

$$\operatorname{argmax}_{\omega \in \Omega} \left| \sum_{\phi \in \Phi} \bar{\kappa}_\phi \phi(\omega) \right|$$

Row generation: *adding constraints* back intelligently

Remove

$$\sum_{\phi \in \Phi} \kappa_\phi \phi \gtreqless 0$$

(this relaxes the problem)

Iteratively, add back

$$\sum_{\phi \in \Phi} \kappa_\phi \phi(\omega) \gtreqless 0$$

for some wisely chosen ω in Ω:

- Each iteration, a solution $\bar{\kappa}$ is obtained;
  choose the next ω such that $\bar{\kappa}$ violates the corresponding constraint.
- A 'deep cut' is generated by solving the problem

$$\mathrm{argmax}_{\omega \in \Omega} \left| \sum_{\phi \in \Phi} \bar{\kappa}_\phi \phi(\omega) \right|$$

- Iteration stops when no feasible $\bar{\kappa}$ or no violated constraint ω can be found.

Propositional sentences

Combinations of $\{0,1\}$-valued 'literals' β_ℓ with ℓ in a finite index set $\mathcal{L}$

Operations: disjunction $\vee$, conjunction $\wedge$, and the negation $\neg$

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Example: $\varphi(\beta) \coloneqq (\beta_\spadesuit \vee \neg\beta_\clubsuit) \wedge \beta_\heartsuit \quad \text{with} \quad \mathcal{L} \coloneqq \{\spadesuit, \clubsuit, \heartsuit\}$

Propositional sentences represent events

Combinations of $\{0,1\}$-valued 'literals' β_ℓ with ℓ in a finite index set $\mathcal{L}$

Operations: disjunction $\vee$, conjunction $\wedge$, and the negation $\neg$

Example: $\varphi(\beta) \coloneqq (\beta_\spadesuit \vee \neg\beta_\clubsuit) \wedge \beta_\heartsuit \quad \text{with} \quad \mathcal{L} \coloneqq \{\spadesuit, \clubsuit, \heartsuit\}$

Possibility space:

$$\Omega_\Gamma \coloneqq \{\beta \in \{0,1\}^{\mathcal{L}} : \chi_\Gamma(\beta) = 1\}$$

$$\Omega_\Delta \coloneqq \{\beta \in \{0,1\}^{\mathcal{L}} : \chi_\Delta(\beta) = 1\}$$

Propositional sentences as basic functions

Combinations of $\{0,1\}$-valued 'literals' β_ℓ with ℓ in a finite index set $\mathcal{L}$

Operations: disjunction $\vee$, conjunction $\wedge$, and the negation $\neg$

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Possibility space:

$$\Omega_\Gamma \coloneqq \{\beta \in \{0,1\}^{\mathcal{L}} : \chi_\Gamma(\beta) = 1\}$$

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Basic functions are indicators of events:

$$\Phi \subset \{0,1\}^{\mathcal{L}} \to \{0,1\}$$

Assume, w.l.o.g., that $\phi(\beta) = \beta_\phi$ for all ϕ in Φ.

Row generation in a propositional context

Form of constraints to generate:

$$\sum_{\phi \in \Phi} \kappa_\phi \bar{\beta}_\phi \lesseqqgtr 0$$

So truth assignments $\bar{\beta}$ in $\{0,1\}^{\mathcal{L}}$ must be generated.

Row generation in a propositional context: any cut

Form of constraints to generate:

$$\sum_{\phi \in \Phi} \kappa_\phi \bar{\beta}_\phi \gtreqless 0$$

So truth assignments $\bar{\beta}$ in $\{0,1\}^{\mathcal{L}}$ must be generated.

By solving a SAT instance:

$$\begin{array}{rl} \text{find} & \beta \in \{0,1\}^{\mathcal{L}} \\ \text{such that} & \chi_\Gamma(\beta) = 1 \end{array} \quad \text{or} \quad \begin{array}{rl} \text{find} & \beta \in \{0,1\}^{\mathcal{L}} \\ \text{such that} & \chi_\Delta(\beta) = 1 \end{array}$$

Row generation in a propositional context: deep cuts

Form of constraints to generate:

$$\sum_{\phi \in \Phi} \kappa_\phi \bar{\beta}_\phi \lesseqqgtr 0$$

So truth assignments $\bar{\beta}$ in $\{0,1\}^{\mathcal{L}}$ must be generated.

By solving a SAT instance:

$$\begin{aligned} \text{find} \quad & \beta \in \{0,1\}^{\mathcal{L}} \\ \text{such that} \quad & \chi_\Gamma(\beta) = 1 \end{aligned} \qquad \text{or} \qquad \begin{aligned} \text{find} \quad & \beta \in \{0,1\}^{\mathcal{L}} \\ \text{such that} \quad & \chi_\Delta(\beta) = 1 \end{aligned}$$

By solving a WPMaxSAT instance:

$$\begin{aligned} \text{maximize} \quad & \sum_{\phi \in \Phi} \bar{\kappa}_\phi \beta_\phi \\ \text{subject to} \quad & \beta \in \{0,1\}^{\mathcal{L}} \\ & \chi_\Gamma(\beta) = 1 \end{aligned} \qquad \text{or} \qquad \begin{aligned} \text{minimize} \quad & \sum_{\phi \in \Phi} \bar{\kappa}_\phi \beta_\phi \\ \text{subject to} \quad & \beta \in \{0,1\}^{\mathcal{L}} \\ & \chi_\Delta(\beta) = 1 \end{aligned}$$

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> ***No***. The original problem is not well-posed. ***Stop***.

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Init. Set $i := 0$. Is χ_Γ or χ_Δ satisfiable?

No. The original problem is not well-posed. ***Stop***.

Yes.
- If $\bar{\gamma}$ satisfies χ_Γ, set $\Gamma_0 := \{\bar{\gamma}\}$; otherwise set $\Gamma_0 := \varnothing$ and $\bar{\gamma} := 0$.
- If $\bar{\delta}$ satisfies χ_Δ, set $\Delta_0 := \{\bar{\delta}\}$; otherwise set $\Delta_0 := \varnothing$ and $\bar{\delta} := 0$.

The *propositional* CONEstrip algorithm

Iter. *Master problem:*
Does the linear program below have a solution $(\bar{\lambda}, \bar{\mu}, \bar{\sigma}, \bar{\kappa})$?

max. $\sum_{\mathcal{D} \in \mathcal{R}_i} \lambda_{\mathcal{D}}$

s.t. $\lambda_{\mathcal{D}} \in [0,1]$ and $\mu \in (\mathbb{R}_{\geq 0})^{\mathcal{D}}$ for all $\mathcal{D}$ in $\mathcal{R}_i$ and $\sigma \in \mathbb{R}_{\geq 1}$

$\sum_{\mathcal{D} \in \mathcal{R}_i} \lambda_{\mathcal{D}} \geq 1$ and $\begin{cases} \sum_{\phi \in \Phi} \kappa_\phi \bar{\beta}_\phi \leq 0 \text{ for all } \bar{\beta} \text{ in } \Gamma_i \\ \sum_{\phi \in \Phi} \kappa_\phi \bar{\beta}_\phi \geq 0 \text{ for all } \bar{\beta} \text{ in } \Delta_i \end{cases}$

$\lambda_{\mathcal{D}} \leq \mu_{\mathcal{D}}$ for all $\mathcal{D}$ in $\mathcal{R}_i$

wh. $\kappa_\phi := \left(\sum_{\mathcal{D} \in \mathcal{R}_0} \sum_{g \in \mathcal{D}} \mu_{\mathcal{D},g} g_\phi\right) - \sigma f_\phi \in \mathbb{R}$ for all ϕ in Φ.

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No. $f \notin \underline{\mathcal{R}_0}$. ***Stop***.

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No. $f \notin \underline{\mathcal{R}_0}$. ***Stop***.

Yes. Let $\mathcal{Q} := \{\mathcal{D} \in \mathcal{R}_i : \bar{\lambda}_{\mathcal{D}} = 0\}$ and set $\mathcal{R}_{i+1} := \mathcal{R}_i \setminus \mathcal{Q}$.

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Iter. *Row generation:*

- If $\Gamma_i \neq \varnothing$, let $\bar{\gamma}$ be a maximizer of $\sum_{\phi \in \Phi} \bar{\kappa}_\phi \beta_\phi$ under χ_Γ; set $\Gamma_{i+1} \coloneqq \Gamma_i \cup \{\bar{\gamma}\}$.
- If $\Delta_i \neq \varnothing$, let $\bar{\delta}$ be a minimizer of $\sum_{\phi \in \Phi} \bar{\kappa}_\phi \beta_\phi$ under χ_Δ; set $\Delta_{i+1} \coloneqq \Delta_i \cup \{\bar{\delta}\}$.

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Is $\sum_{\phi\in\Phi} \bar{\kappa}_\phi \bar{\gamma}_\phi \leq 0 \leq \sum_{\phi\in\Phi} \bar{\kappa}_\phi \bar{\delta}_\phi$ and $\{\mathcal{D} \in \mathcal{Q} \colon \bar{\mu}_\mathcal{D} = 0\} = \mathcal{Q}$?

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Yes. Set $t \coloneqq i + 1$; $f \in \underline{\mathcal{R}_t} \subseteq \underline{\mathcal{R}_0}$. ***Stop***.

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Yes. Set $t \coloneqq i+1$; $f \in \underline{\mathcal{R}_t} \subseteq \underline{\mathcal{R}_0}$. ***Stop.***

No. Increase i's value by 1. ***Reiterate.***

Thank you!

Questions?