Modeling **Risk & Uncertainty** using **Accept & Reject Statements**

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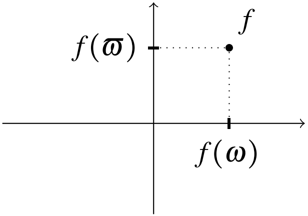
The setup

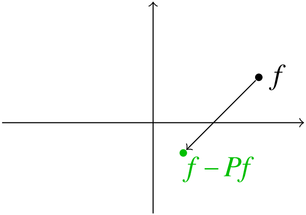
- Experiment with outcomes in some possibility space Ω.
- Agent uncertain about the experiment's outcome.
- Linear space $\mathcal{L}$ of real-valued gambles on Ω.



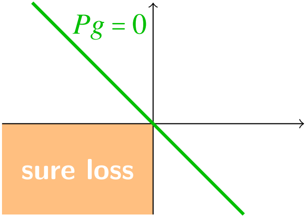
- Agent expresses uncertainty by making statements about gambles, forming an assessment.
- Agent wishes to rationally deduce inferences and draw conclusions from this assessment.

The work we build on

- De Finetti: previsions P.



$\Rightarrow$



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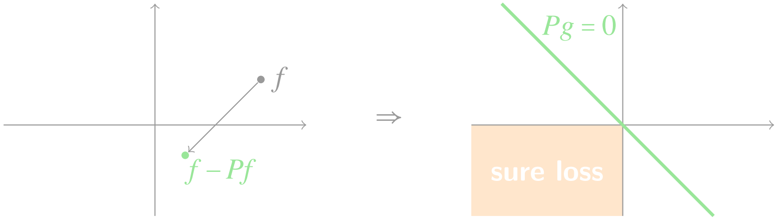
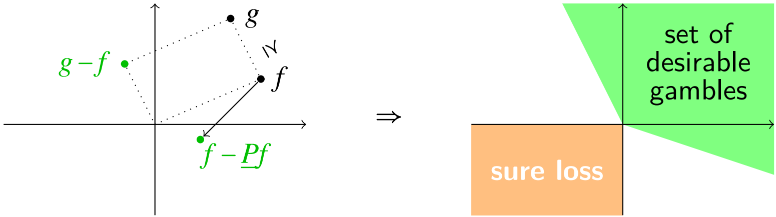


- Williams, Seidenfeld et al., Walley:
    - lower previsions $\underline{P}$,
    - sets of acceptable/favorable/desirable gambles,
    - partial preference orders $\geq$.



Accepting & Rejecting Gambles

Accepting a gamble f implies a commitment to engage in the following transaction:

(i) the experiment's outcome $\omega \in \Omega$ is determined,

(ii) the agent gets the—possibly negative—payoff $f(\omega)$.

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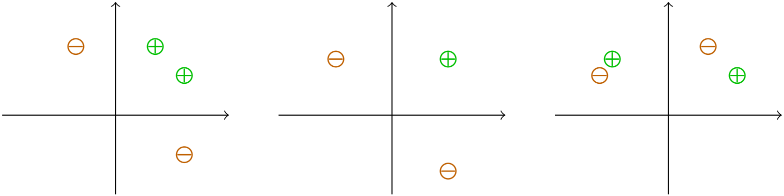
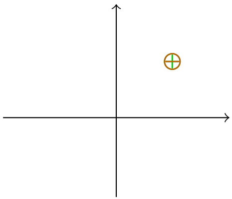
Assessment A pair $\mathcal{A} \coloneqq \langle \mathcal{A}_{\geq} ; \mathcal{A}_{\prec} \rangle$ of sets of accepted and rejected gambles.



Gamble Categorization

Accepted	$\mathcal{A}_{\geq}$.
Rejected	$\mathcal{A}_{\prec}$.
Unresolved	Neither accepted nor rejected; $\mathcal{A}_{\smile} \coloneqq \mathcal{L} \setminus (\mathcal{A}_{\geq} \cup \mathcal{A}_{\prec})$.
Confusing	Both accepted and rejected; $\mathcal{A}_{\geq,\prec} \coloneqq \mathcal{A}_{\geq} \cap \mathcal{A}_{\prec}$.



Gamble Categorization

Accepted $\mathcal{A}_{\succeq}$.

Rejected $\mathcal{A}_{\prec}$.

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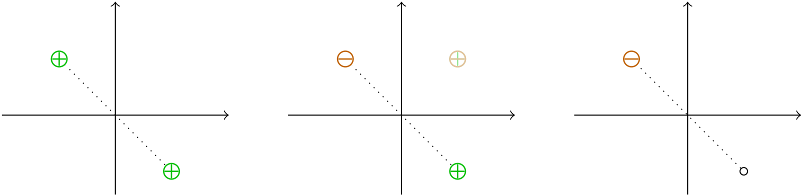

Indifferent Both it and its negation accepted; $\mathcal{A}_{\simeq} \coloneqq \mathcal{A}_{\succeq} \cap -\mathcal{A}_{\succeq}$.

Favorable Accepted with a rejected negation; $\mathcal{A}_{\succ} \coloneqq \mathcal{A}_{\succeq} \cap -\mathcal{A}_{\prec}$.

Indeterminate Both it and its negation not acceptable; $(\mathcal{A}_{\parallel} \coloneqq \mathcal{A}_{\succeq} \cup -\mathcal{A}_{\succeq})^c$.

Axiom: **No Confusion**

Because of the interpretation attached to acceptance and rejection statements, we consider confusion irrational.

So we require assessments $\mathcal{A}$ to not contain confusion:

$$\mathcal{A}_{\geq,\prec} = \mathcal{A}_{\geq} \cap \mathcal{A}_{\prec} = \emptyset$$

Axiom *template*: **Background Model**

Problem domain specific set of acceptable gambles $\mathcal{S}_{\geq}$ and set of rejected gambles $\mathcal{S}_{<}$. To be combined with the agent's own assessment.

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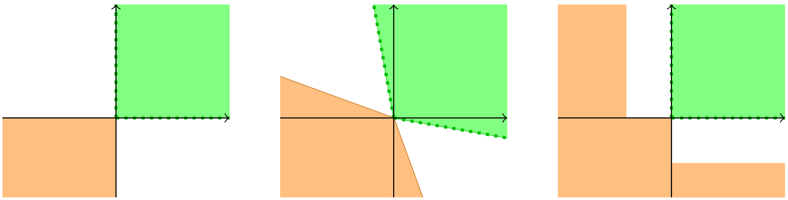

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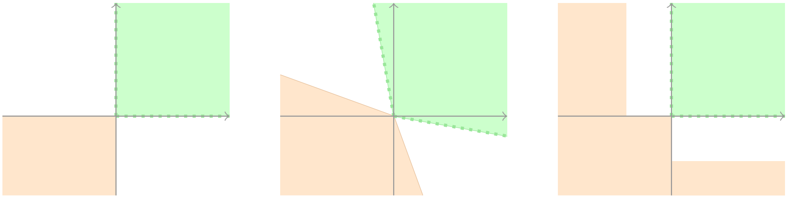

For convenience, assume Indifference to Status Quo: $0 \in \mathcal{S}_{\geq}$.

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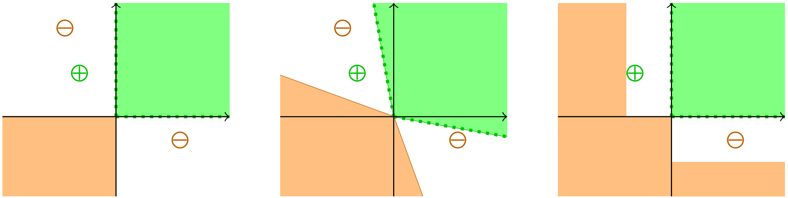

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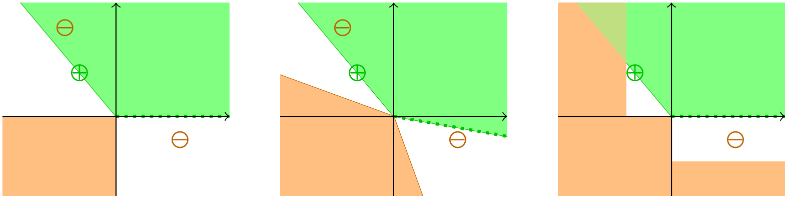

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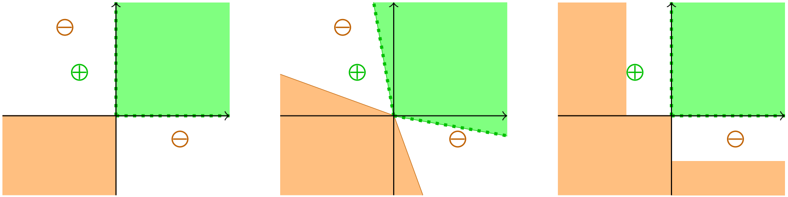

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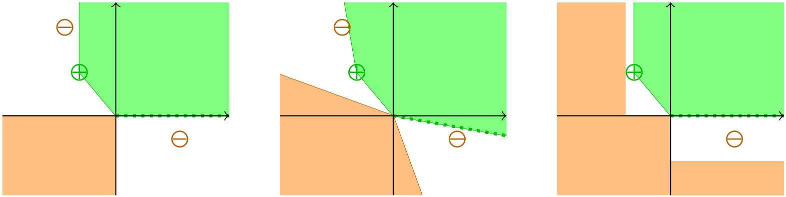

Axiom *template*: **Deductive Closure**

An assessment $\mathcal{A}$ can be deductively extended to a deductively closed assessment $\mathcal{D}$;

1. $\mathcal{D} \coloneqq \langle \operatorname{posi} \mathcal{A}_{\geq} ; \mathcal{A}_{\prec} \rangle$,
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The assumptions underlying the choice of a deductive extension rule lead us to exclusively use deductively closed assessments $\mathcal{D}$ for inference and decision purposes:

1. $\operatorname{posi} \mathcal{D}_{\geq} = \mathcal{D}_{\geq}$
2. $\operatorname{co} \mathcal{D}_{\geq} = \mathcal{D}_{\geq}$

Gambles in limbo & reckoning extension

Deductive Closure interacts with No Confusion:

- Consider a deductively closed assessment $\mathcal{D}$.
- Additionally consider some unresolved gamble f acceptable.
- Apply deductive extension to $\langle \mathcal{D}_{\succeq} \cup \{f\}; \mathcal{D}_{\prec} \rangle$.
- For some f, this would lead to an increase in confusion.
- These have the same effect as gambles in $\mathcal{D}_{\prec}$,
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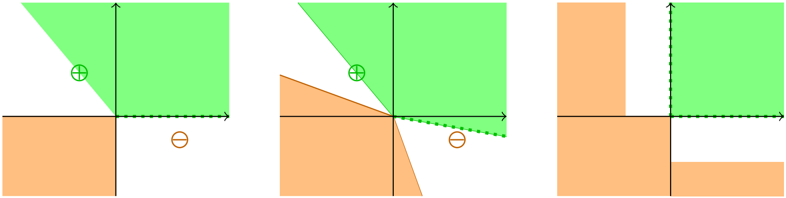

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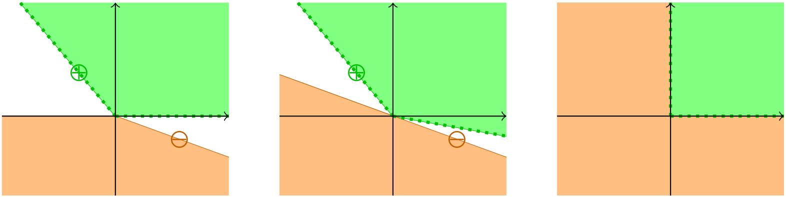

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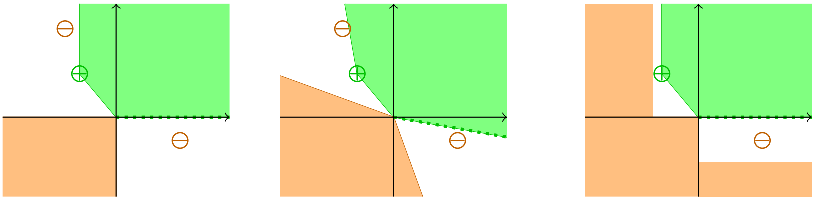

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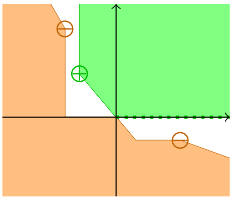
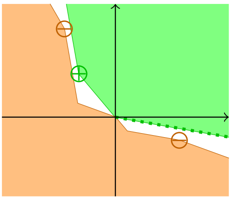
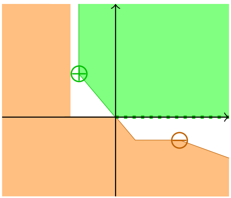

Axiom: **No Limbo**

We consider accepting gambles in limbo unreasonable and therefore further restrict attention to models $\mathcal{M}$ for inference and decision purposes:

1. $\{\mu\mathcal{M}_{\prec} - \mathcal{M}_{\geq} : \mu > 0\} \subseteq \mathcal{M}_{\prec}$
2. $\{\mu\mathcal{M}_{\prec} - \frac{\mu-1}{\mu}\mathcal{M}_{\geq} : \mu \geq 1\} \subseteq \mathcal{M}_{\prec}$

Conclusions

- Our framework further generalizes existing generalizations of probability theory.
- The generalization is flexible and allows for interesting model types.
- The mathematics do not become much more complex.
- There is another, as of yet unmentioned interesting aspect: combination of nonstrict and strict preference orders.

Want to know more: read the full paper!

http://arxiv.org/abs/1208.4462