Modeling Risk & Uncertainty using Accept & Reject Statements

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The setup

- Experiment with outcomes in some possibility space $\Omega$.
- Agent uncertain about the experiment’s outcome.
- Linear space $\mathcal{L}$ of real-valued gambles on $\Omega$.

- Agent expresses uncertainty by making statements about gambles, forming an assessment.
- Agent wishes to rationally deduce inferences and draw conclusions from this assessment.
The work we build on

- De Finetti: previsions $P$.

\[ f - Pf \Rightarrow \text{sure loss} \]

\[Pg = 0\]
The work we build on

- De Finetti: previsions $P$.

- Williams, Seidenfeld et al., Walley:
  - lower previsions $\underline{P}$,
  - sets of acceptable/favorable/desirable gambles,
  - partial preference orders $\succeq$. 

$$f - Pf$$

$$Pg = 0$$

$$g - f$$

$$f - Pf$$

$\Rightarrow$ sure loss

set of desirable gambles
Accepting & Rejecting Gambles

**Accepting** a gamble $f$ implies a commitment to engage in the following transaction:

(i) the experiment’s outcome $\omega \in \Omega$ is determined,
(ii) the agent gets the—possibly negative—payoff $f(\omega)$.

**Rejecting** a gamble: the agent considers accepting it unreasonable.
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(i) the experiment’s outcome $\omega \in \Omega$ is determined,
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Rejecting a gamble: the agent considers accepting it unreasonable.

Assessment A pair $\mathcal{A} := \langle \mathcal{A}_\geq; \mathcal{A}_\leq \rangle$ of sets of accepted and rejected gambles.
Gamble Categorization

Accepted $\mathcal{A}_\geq$.

Rejected $\mathcal{A}_<$.  

Unresolved Neither accepted nor rejected; $\mathcal{A}_\sim := \mathcal{L} \setminus (\mathcal{A}_\geq \cup \mathcal{A}_<)$.  

Confusing Both accepted and rejected; $\mathcal{A}_{\geq,<} := \mathcal{A}_\geq \cap \mathcal{A}_<$.  

Favorable Accepted with a rejected negation; $\mathcal{A}_{\sim^+} := \mathcal{A}_\geq \cap -\mathcal{A}_<$.  

Indeterminate Both it and its negation not acceptable; $\{\mathcal{A}_{\sim^+} : = \mathcal{A}_\geq \cup -\mathcal{A}_\geq\}$.  

\[ \oplus \quad \ominus \]
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Indifferent Both it and its negation accepted; \( \mathcal{A}_{\sim} := \mathcal{A}_\geq \cap -\mathcal{A}_\geq \).

Favorable Accepted with a rejected negation; \( \mathcal{A}_{\succ} := \mathcal{A}_\geq \cap -\mathcal{A}_< \).

Indeterminate Both it and its negation not acceptable; \( (\mathcal{A}_{\parallel} := \mathcal{A}_\geq \cup -\mathcal{A}_\geq)^c \).
Axiom: No Confusion

Because of the interpretation attached to acceptance and rejection statements, we consider confusion irrational.

So we require assessments $\mathcal{A}$ to not contain confusion:

$$\mathcal{A}_{\geq,<} = \mathcal{A}_{\geq} \cap \mathcal{A}_{<} = \emptyset$$
Axiom *template*: **Background Model**

Problem domain specific set of acceptable gambles $\mathcal{S}_\geq$ and set of rejected gambles $\mathcal{S}_\prec$. To be combined with the agent’s own assessment.
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Axiom *template: Background Model*

Problem domain specific set of acceptable gambles $S_\geq$ and set of rejected gambles $S_\prec$. To be combined with the agent’s own assessment.

For convenience, assume **Indifference to Status Quo**: $0 \in S_\geq$. 
Deductive extension

The nature of the gamble payoffs (utility considerations) determines a deductive extension rule for acceptable gambles: given a set of acceptable gambles, which other gambles should be acceptable to the agent.
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1. **Positive linear combinations** (assumption of linear precise utility):
   - sums of accepted gambles are acceptable.
   - positively scaled accepted gambles are acceptable.
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2. Convex combinations (weakening the assumption of linear precise utility):
   - convex mixtures of accepted gambles are acceptable.
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2. Convex combinations (weakening the assumption of linear precise utility):
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The convex hull operator \( \text{co} \) performs the necessary operation; it generates convex polyhedra.
An assessment $\mathcal{A}$ can be **deductively extended** to a **deductively closed assessment** $\mathcal{D}$;

1. $\mathcal{D} := \langle \text{posi } \mathcal{A}_\geq; \mathcal{A}_\leq \rangle$,

2. $\mathcal{D} := \langle \text{co } \mathcal{A}_\geq; \mathcal{A}_\leq \rangle$. 
Axiom template: Deductive Closure

An assessment \( \mathcal{A} \) can be deductively extended to a deductively closed assessment \( \mathcal{D} \):

1. \( \mathcal{D} := \langle \text{posi } \mathcal{A}_\geq; \mathcal{A}_\leq \rangle \),
2. \( \mathcal{D} := \langle \text{co } \mathcal{A}_\geq; \mathcal{A}_\leq \rangle \).

The assumptions underlying the choice of a deductive extension rule lead us to exclusively use deductively closed assessments \( \mathcal{D} \) for inference and decision purposes:

1. \( \text{posi } \mathcal{D}_\geq = \mathcal{D}_\geq \)
2. \( \text{co } \mathcal{D}_\geq = \mathcal{D}_\geq \)
Gambles in limbo & reckoning extension

Deductive Closure interacts with No Confusion:

- Consider a deductively closed assessment $\mathcal{D}$.
- Additionally consider some unresolved gamble $f$ acceptable.
- Apply deductive extension to $\langle \mathcal{D}_\geq \cup \{f\}; \mathcal{D}_\prec \rangle$.
- For some $f$, this would lead to an increase in confusion.
- These have the same effect as gambles in $\mathcal{D}_\prec$, and form the limbo of $\mathcal{D}$.

We use reckoning extension to reject gambles in limbo and create a model $\mathcal{M}$. 
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- For some $f$, this would lead to an increase in confusion.
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We use reckoning extension to reject gambles in limbo and create a model $\mathcal{M}$. 
Axiom: No Limbo

We consider accepting gambles in limbo unreasonable and therefore further restrict attention to models $\mathcal{M}$ for inference and decision purposes:

1. $\{ \mu \mathcal{M}_< - \mathcal{M}_\geq : \mu > 0 \} \subseteq \mathcal{M}_<$

2. $\{ \mu \mathcal{M}_< - \frac{\mu - 1}{\mu} \mathcal{M}_\geq : \mu \geq 1 \} \subseteq \mathcal{M}_<$
Conclusions

- Our framework further generalizes existing generalizations of probability theory.

- The generalization is flexible and allows for interesting model types.

- The mathematics do not become much more complex.

- There is another, as of yet unmentioned interesting aspect: combination of nonstrict and strict preference orders.
Therefore $D = D = A \neq \text{uni}$. So we need to prove that the poset $\{\} \subseteq A/\text{uni}$. To preserve No Confusion under Condition (18) we have to disallow the latter. Therefore must be part of the background model.

Coherent prevision is an interesting and useful special case. Elsewhere in the literature, only the case $\{\} \subseteq A/\text{uni}$ is considered.

Closed assessment may exhibit an increase in confusion: an unresolved gamble $\mathcal{N}$ and doing basic deductive inference.

The maximal models dominating an assessment can also be used for inference purposes: indeed, for any gamble $f$, the maximal models respecting the background model are incomparable when neither of their differences is resolved: $\mathcal{N} \subseteq A/\text{uni}$.

Given $(iii) \cap \mathcal{N} = \text{uni}$. Moreover, violation of inf $\mathcal{N} = \text{uni}$ and because of this are conjugate to $\mathcal{N} = \text{uni}$. Given $\mathcal{N}$, which always includes $\mathcal{N} = \text{uni}$.

The illustration of the nine-element partition of gamble space corresponds to a model $\mathcal{N} = \text{uni}$ and its top $\mathcal{N} = \text{uni}$ returns $\mathcal{O}$. (AR1) is equivalent to $\mathcal{N} + \mathcal{N} = \text{uni}$. Macmillan.

Our models, so that we may also know what type of information they cannot represent. By construction, $\mathcal{N} = \text{uni}$ and taking $\mathcal{N} = \text{uni}$.

Proposition 33: Given $\mathcal{N} = \text{uni}$, then $\mathcal{N} = \text{uni}$.

Hammer (1955, Corollary 2), where $\mathcal{N} = \text{uni}$ and $\mathcal{N} = \text{uni}$.

Given $\mathcal{N} = \text{uni}$, we mean unconfused ones.) The agent provides an assessment that $\mathcal{N} = \text{uni}$ respects the background model.

The unconfused models of a favourability framework with a fixed background model actually form a poset $\{\} \subseteq A/\text{uni}$.

Given $\mathcal{N} = \text{uni}$, and doing basic deductive inference.

2.8. Positing a Background Model

Closed assessments may not be compatible with the background model. Additional background information is needed to determine which assessment is coherent with the background model. In such cases, the assessment can be replaced by a more comprehensive one.

Proposition 34: Given $\mathcal{N} = \text{uni}$, then $\mathcal{N} = \text{uni}$.

Formula (18) implies $\mathcal{N} = \text{uni}$. The agent uses $\mathcal{N} = \text{uni}$ and makes a statement about the gamble to $\mathcal{N} = \text{uni}$.

Theorem 35: Given $\mathcal{N} = \text{uni}$, then $\mathcal{N} = \text{uni}$.

Conclusion: Given $\mathcal{N} = \text{uni}$, then $\mathcal{N} = \text{uni}$.

Want to know more: read the full paper!

http://arxiv.org/abs/1208.4462