Erik
Quaeghebeur

SYSTeMS
Ghent University

Gert
de Cooman

Erik
Quaeghebeur

SYSTeMS
Ghent University

Gert
de Cooman

Arthur
Van Camp

Jasper
De Bock

Stavros
Lopatatzidis

Erik
Quaeghebeur

SYSTeMS
Ghent University

Gert
de Cooman

Arthur
Van Camp

Jasper
De Bock

Stavros
Lopatatzidis

Erik
Quaeghebeur

Decision Support Systems
Utrecht University

Linda
van der Gaag

SYSTeMS
Ghent University

Gert
de Cooman

Arthur
Van Camp

Jasper
De Bock

Stavros
Lopatatzidis

Erik
Quaeghebeur

Decision Support Systems
Utrecht University

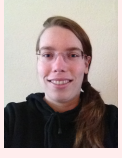
Linda
van der Gaag

Silja
Renooij

Marjan
van den Akker

Steven
Woudenberg

Merel
Rietbergen



SYSTeMS
Ghent University

Gert
de Cooman

Arthur
Van Camp

Jasper
De Bock

Stavros
Lopatatzidis

Erik
Quaeghebeur

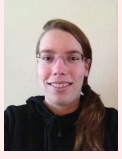
Decision Support Systems
Utrecht University

Linda
van der Gaag

Silja
Renooij

Marjan
van den Akker

Steven
Woudenberg



Merel
Rietbergen

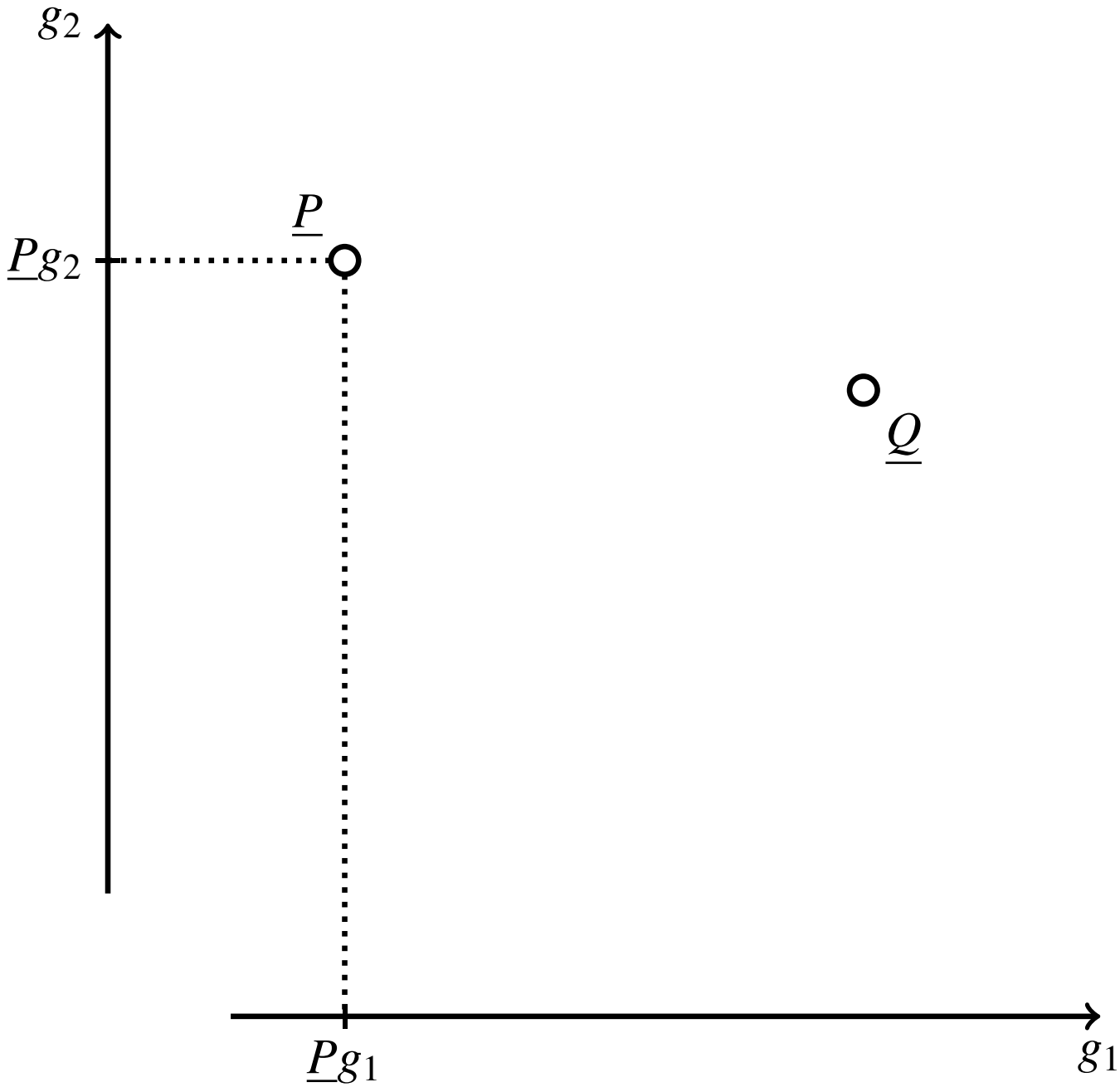
Algorithms & Complexity
Centrum Wiskunde & Informatica

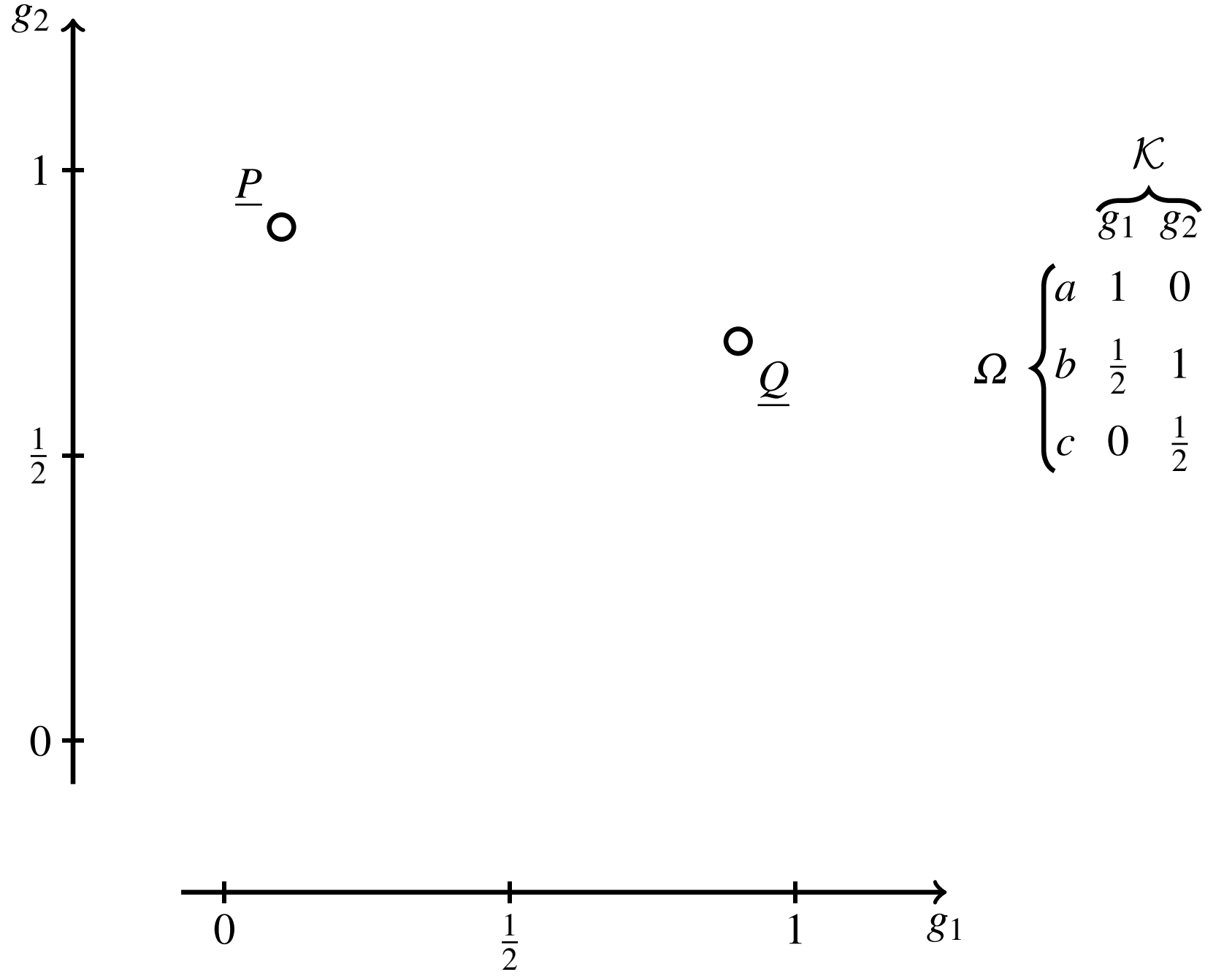
Peter
Grünwald

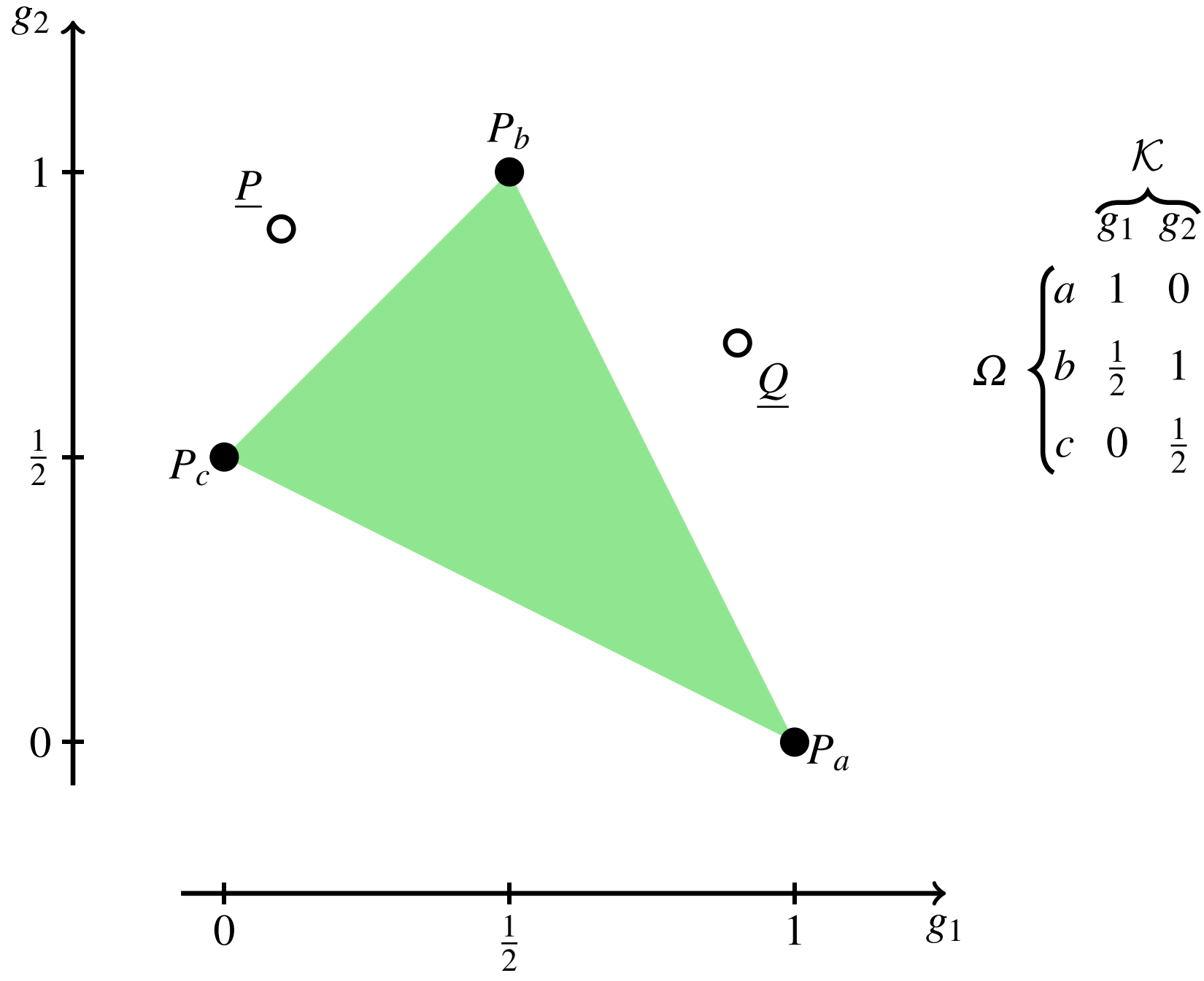
Characterizing Coherence, Correcting Incoherence

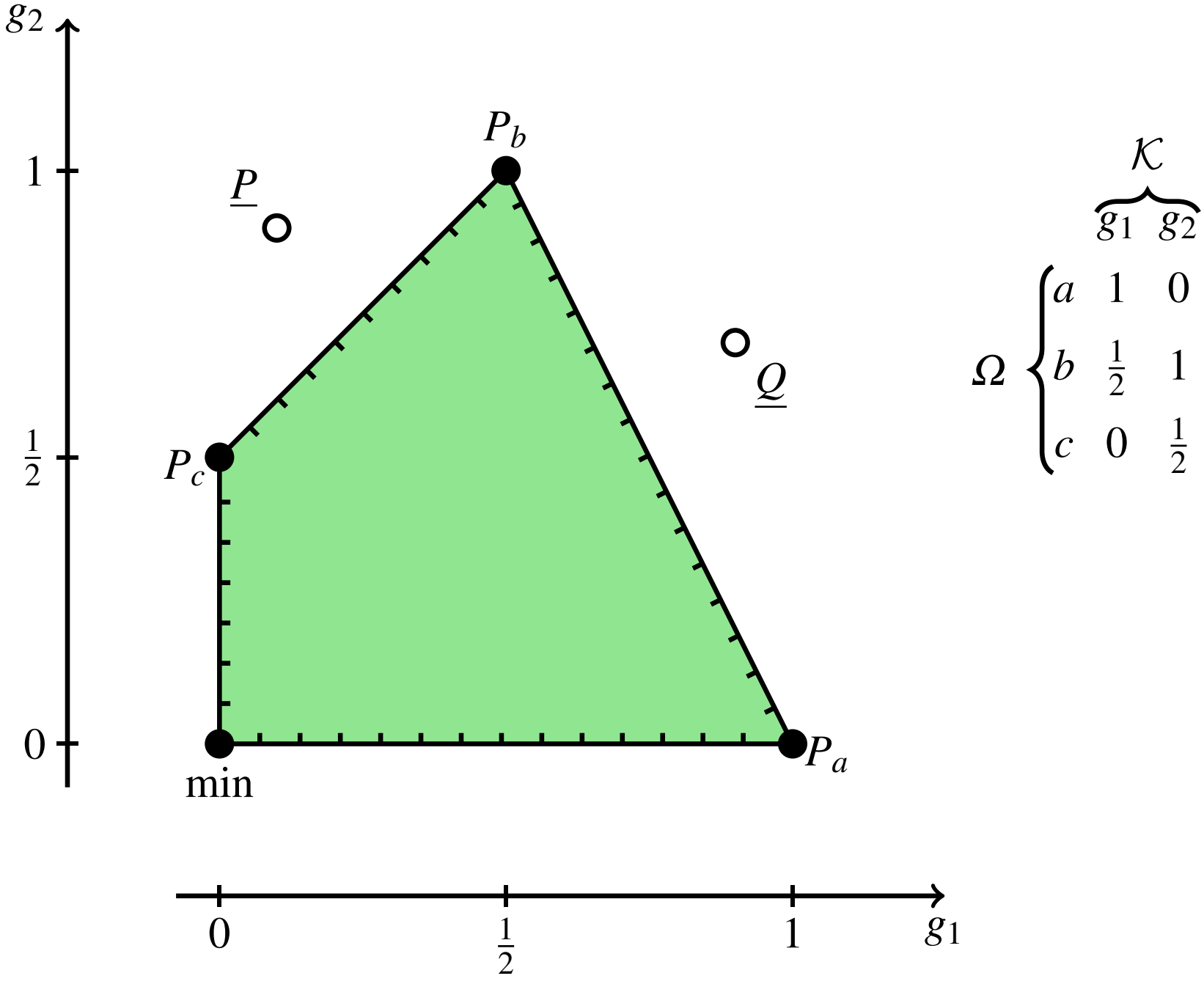
Erik Quaeghebeur

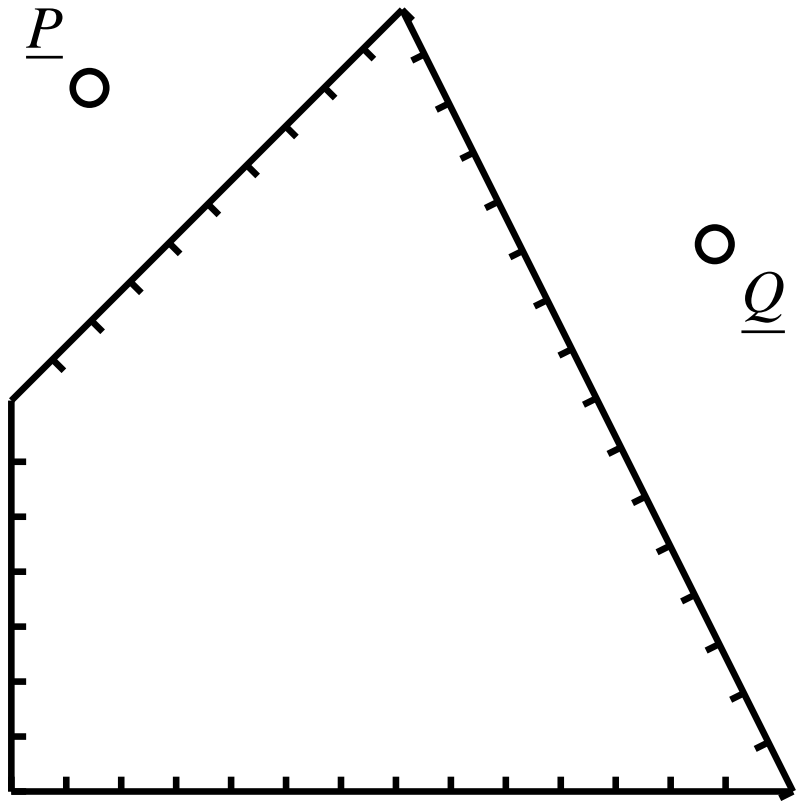

P
Q

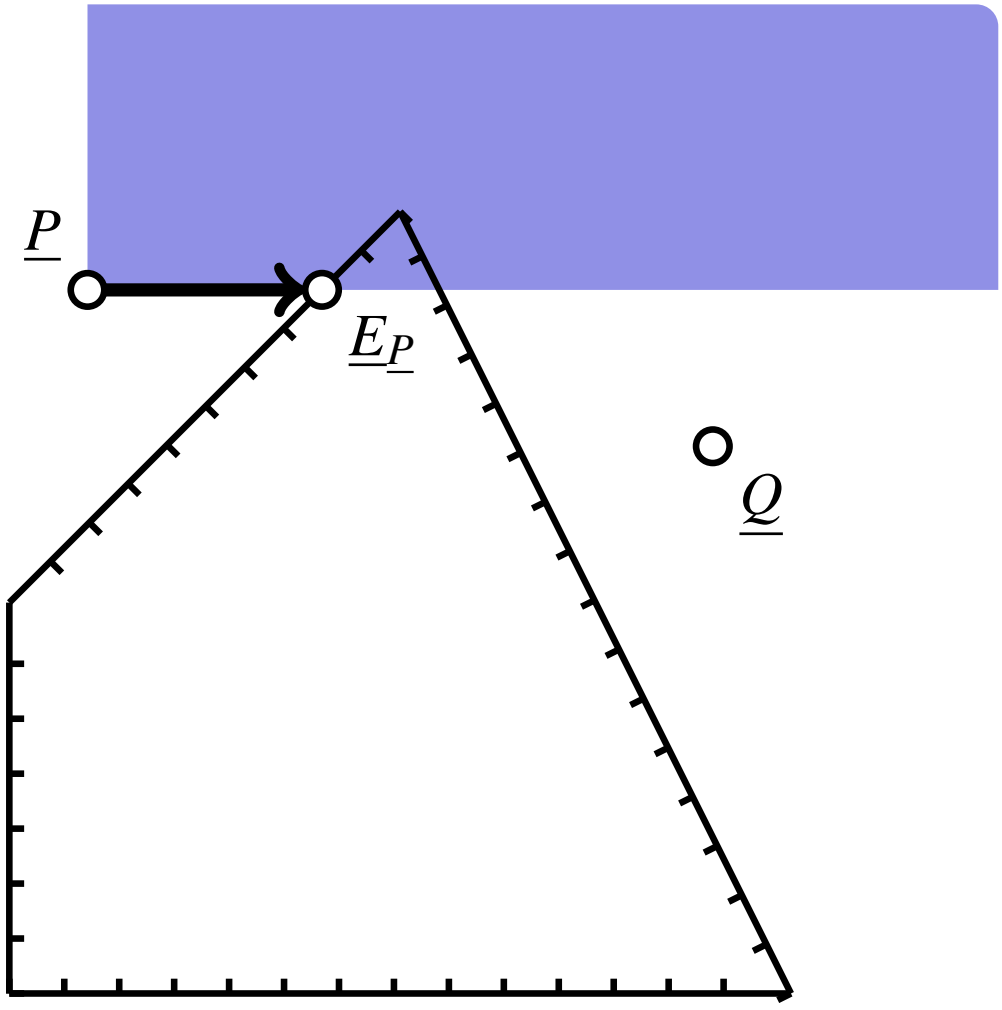

g_2
$\underline{P}$
$\underline{P}g_2$
$\underline{Q}$
$\underline{P}g_1$
g_1

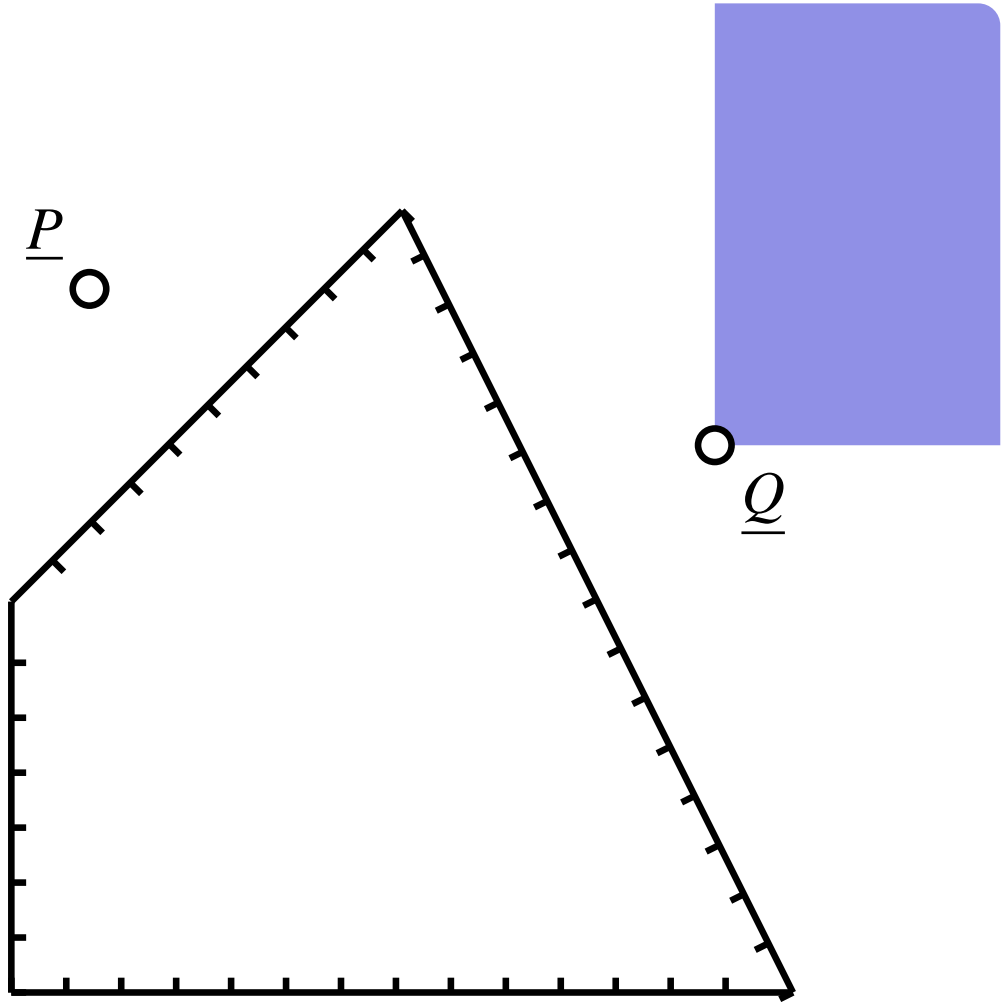

g_2
1
$\frac{1}{2}$
0
P
Q
$\mathcal{K}$
g_1 g_2
Ω
a 1 0
b $\frac{1}{2}$ 1
c 0 $\frac{1}{2}$
0
$\frac{1}{2}$
1
g_1

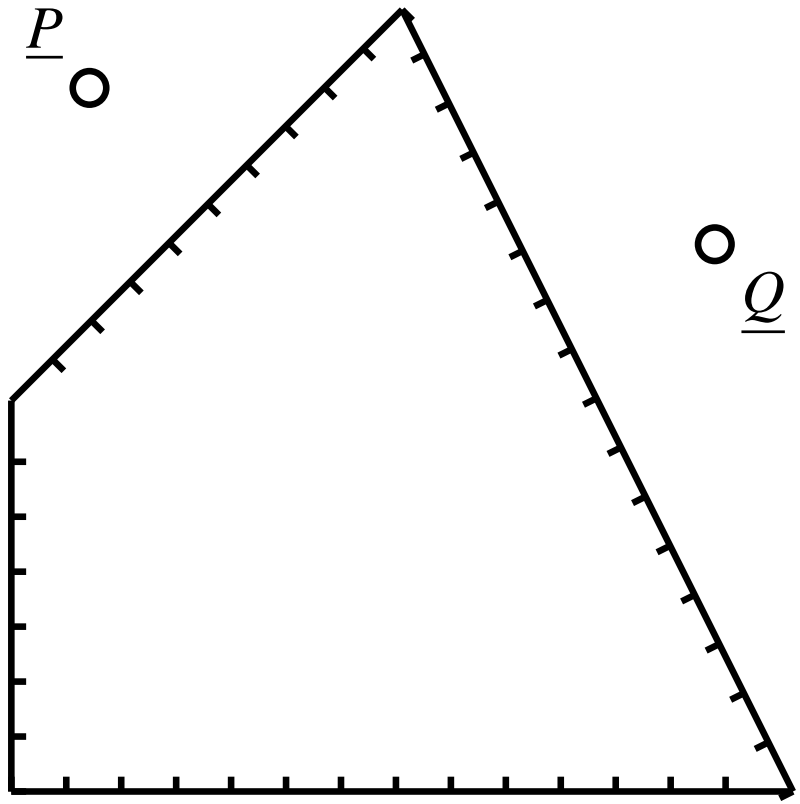

g_2
1
$\frac{1}{2}$
0
P_b
P
Q
P_c
P_a
0
$\frac{1}{2}$
1
g_1
$\mathcal{K}$
g_1 g_2
Ω
a 1 0
b $\frac{1}{2}$ 1
c 0 $\frac{1}{2}$

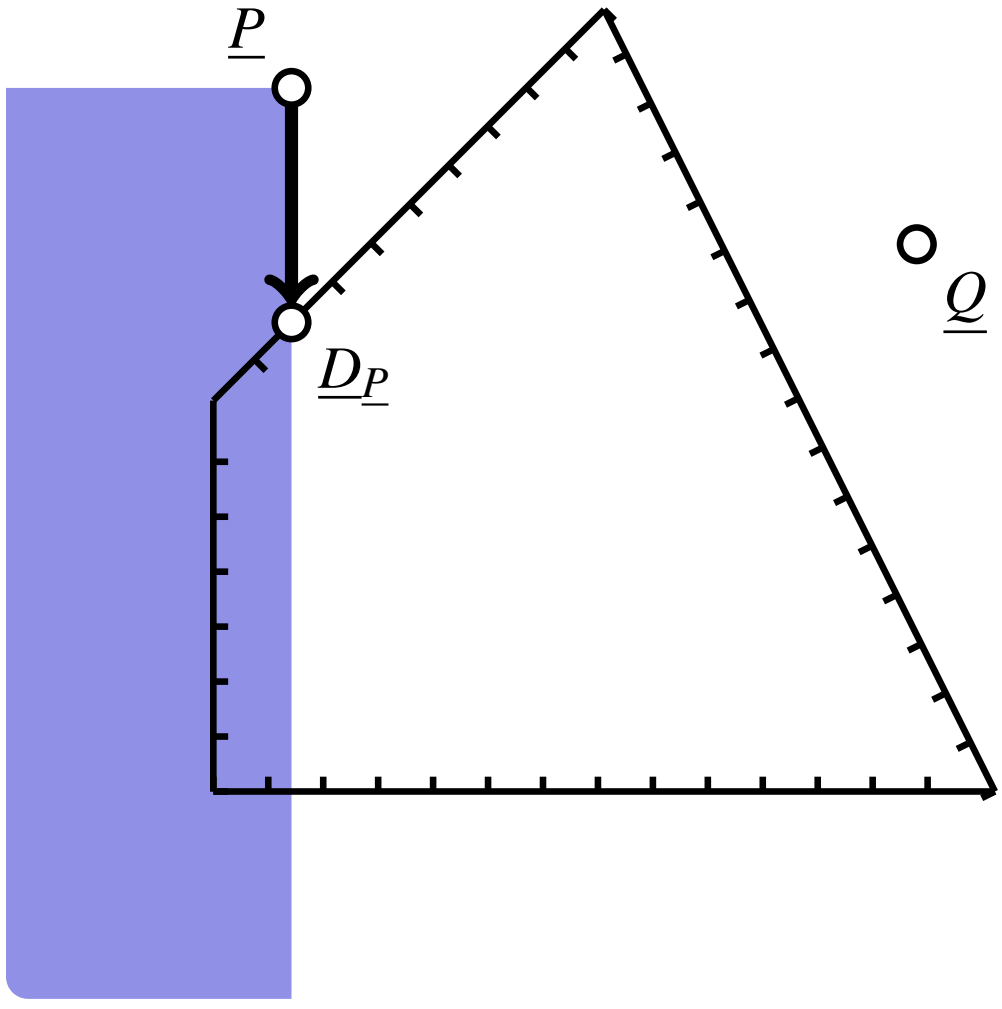

g_2
1
$\frac{1}{2}$
0
P_b
P
Q
P_c
P_a
min
0
$\frac{1}{2}$
1
g_1
$\mathcal{K}$
g_1 g_2
Ω
a 1 0
b $\frac{1}{2}$ 1
c 0 $\frac{1}{2}$

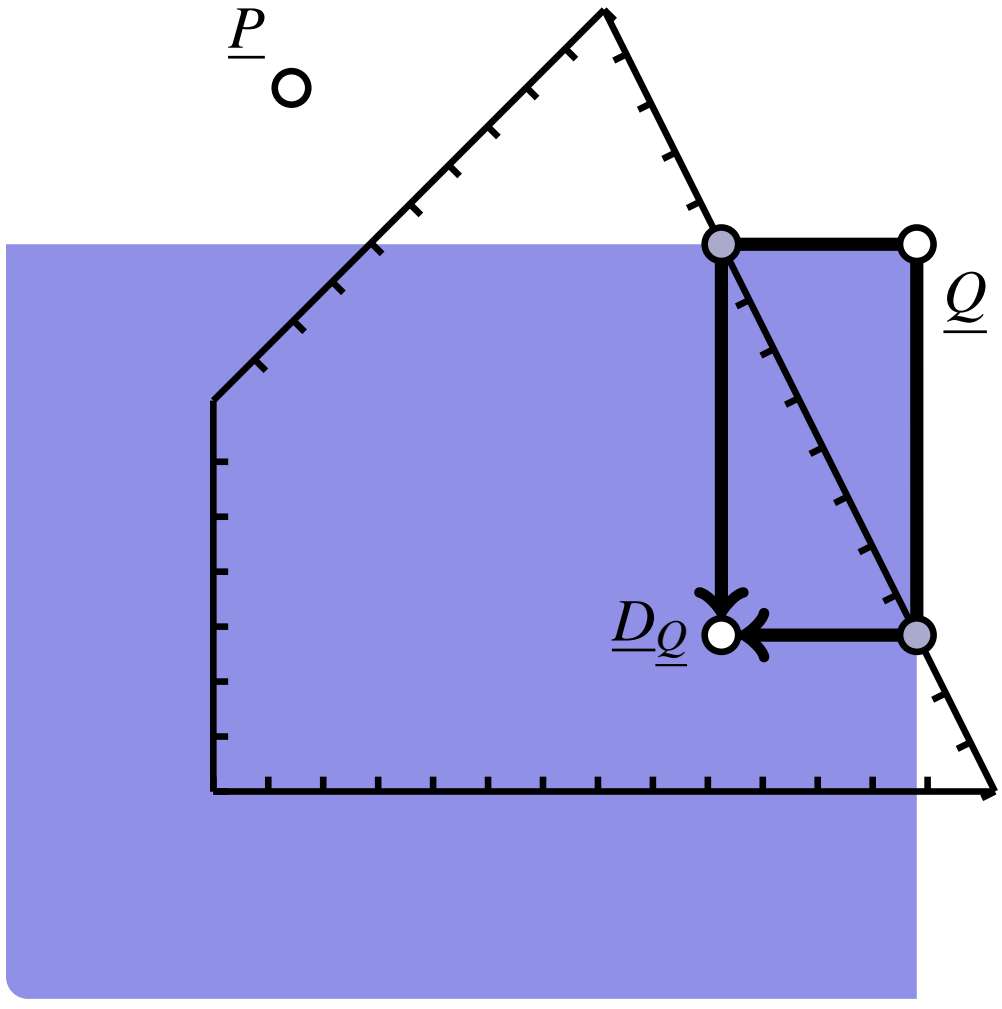

P
Q

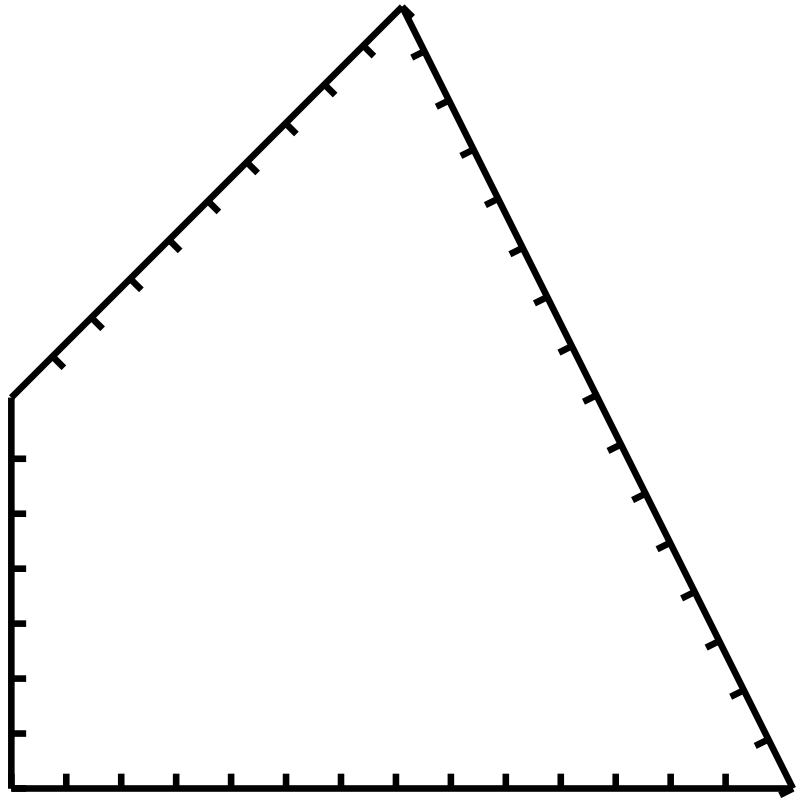

P
E_P
Q

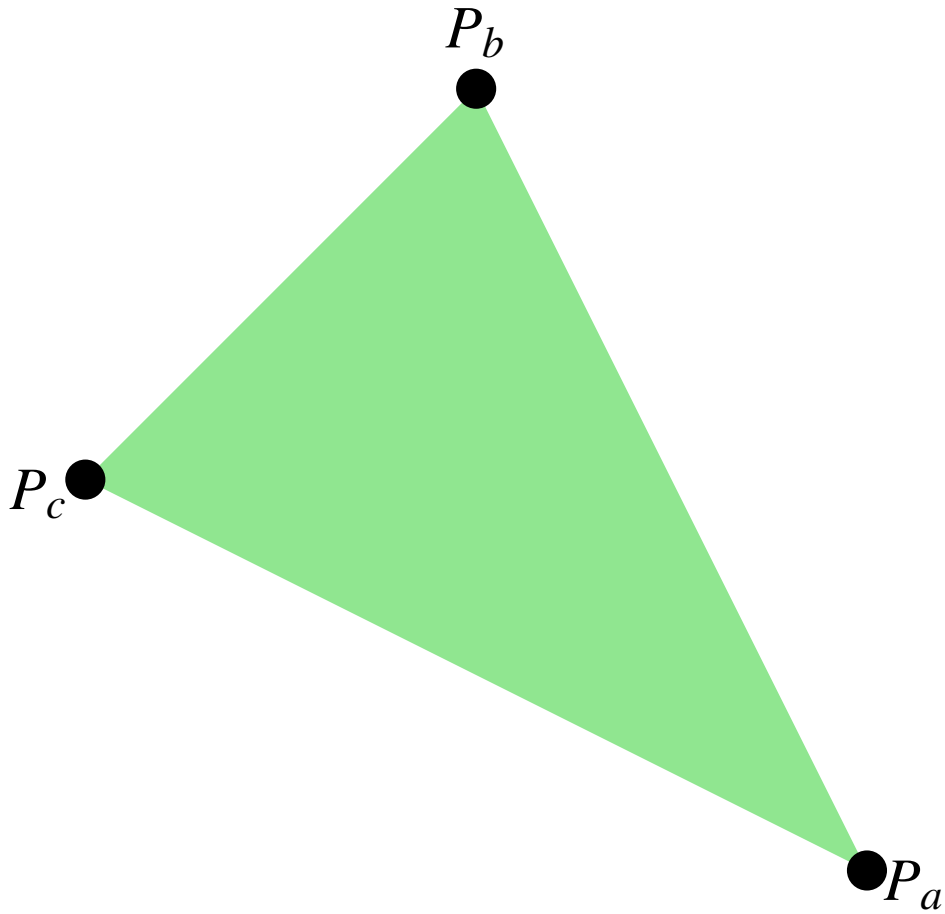

P
Q

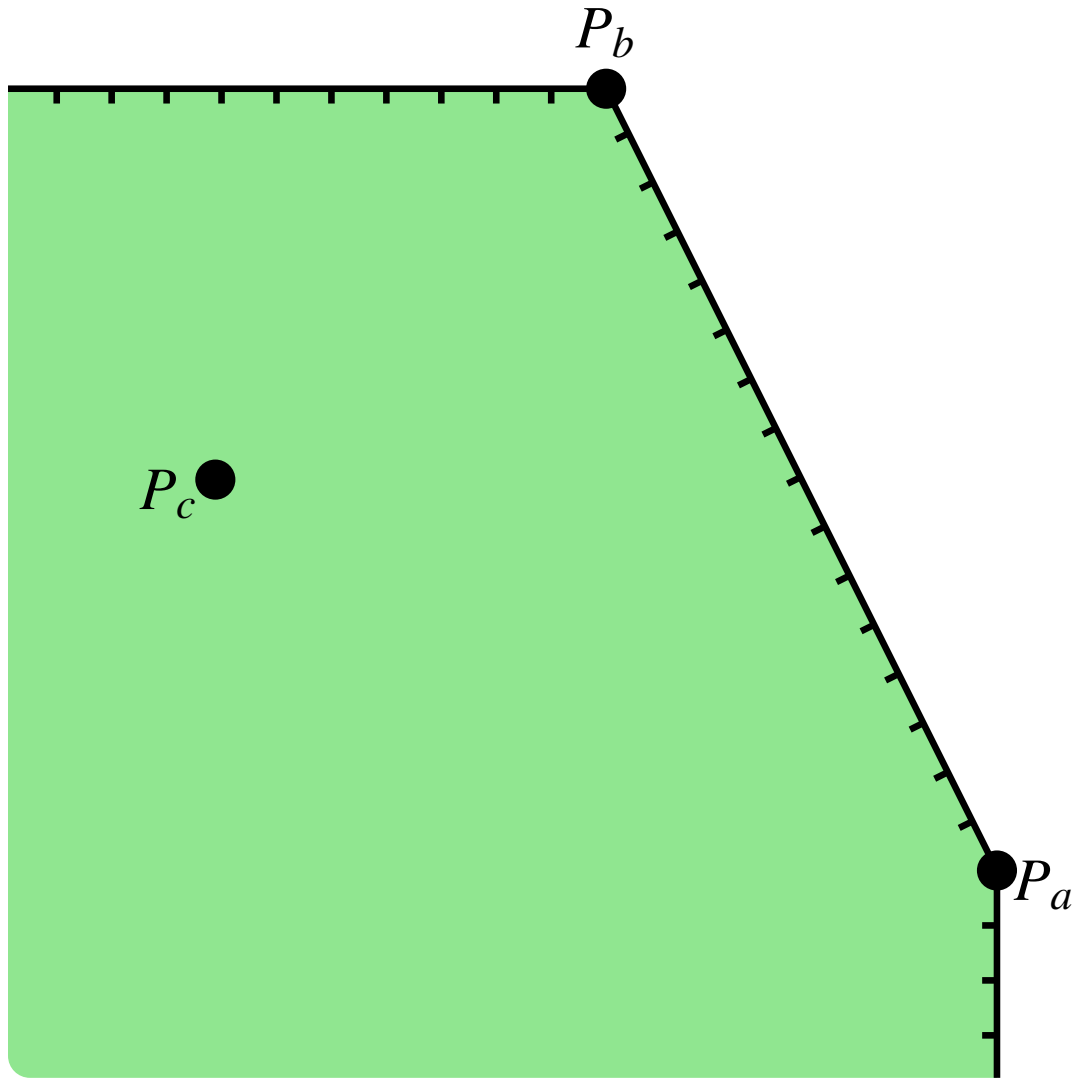

P
Q

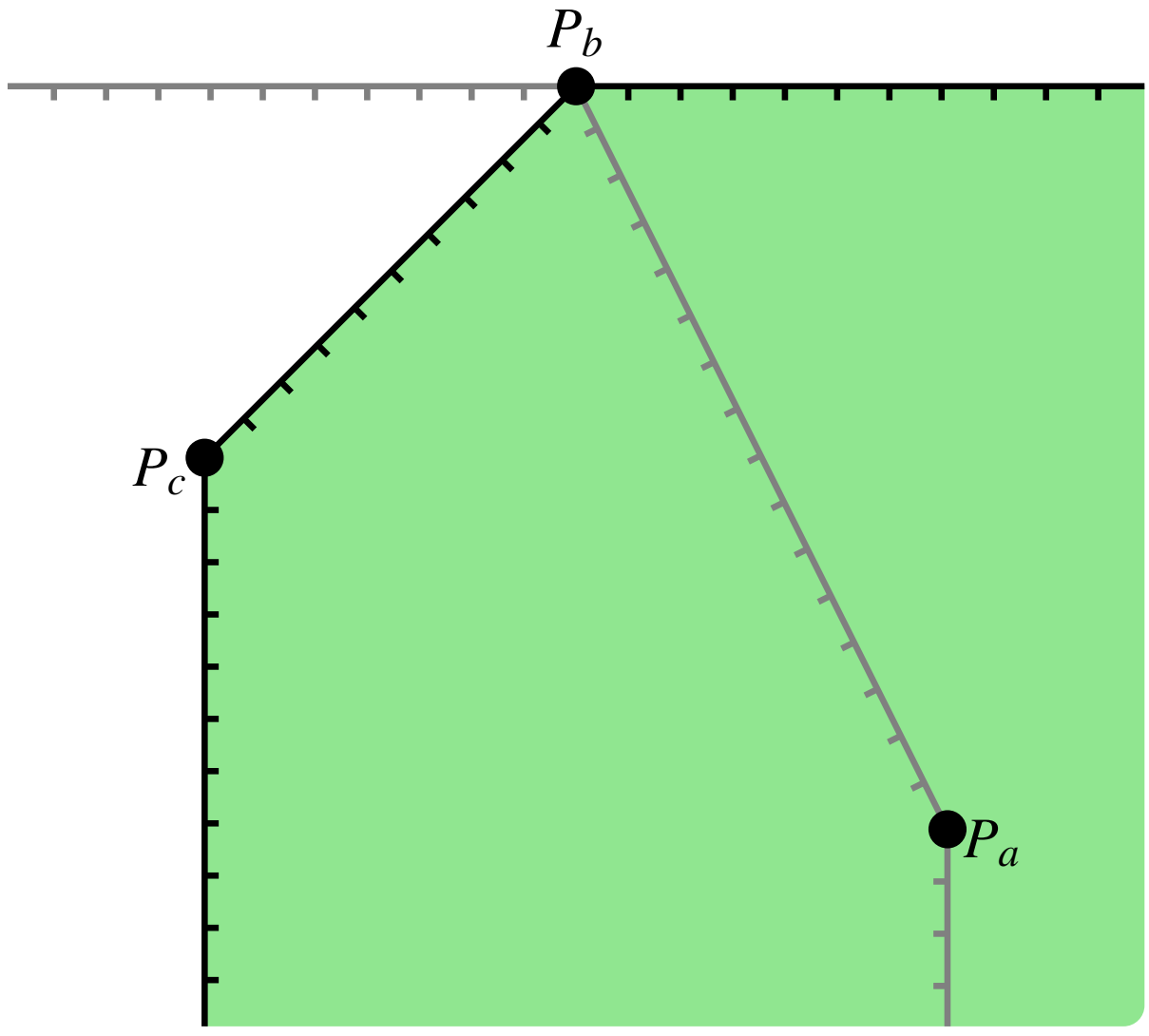

P
Q
D_P

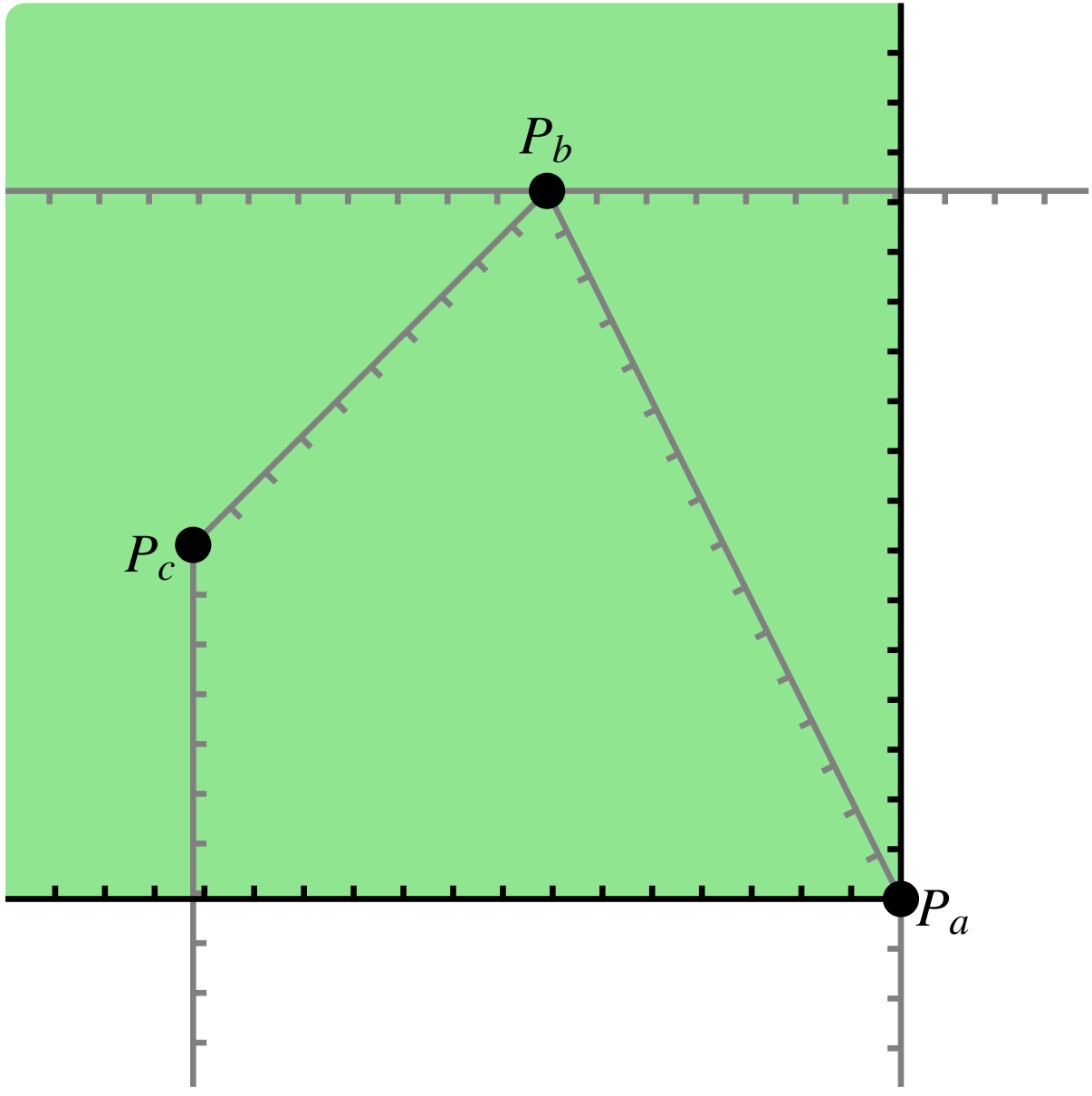

$\underline{P}$
$\underline{Q}$
$D_{\underline{Q}}$

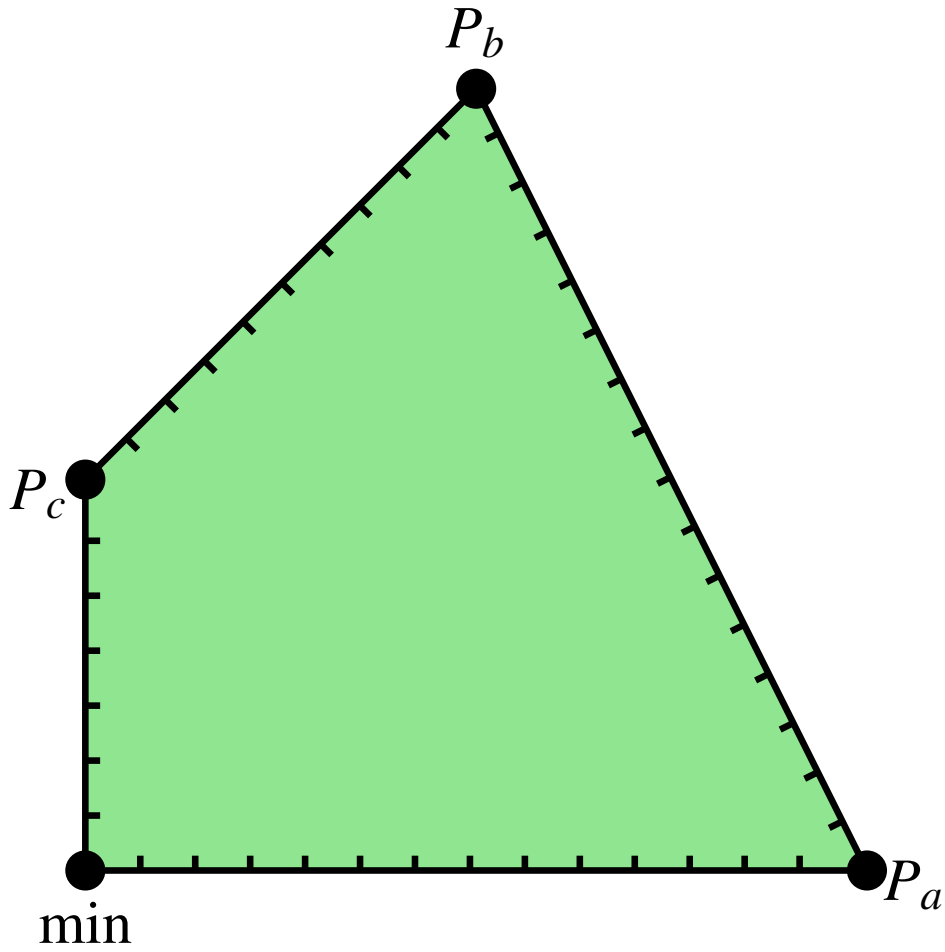



P_b
P_c
P_a


P_b
P_c
P_a


P_b
P_c
P_a


P_b
P_c
P_a


P_b
P_c
P_a
min

Characterizing Coherence, Correcting Incoherence

1. Context

Basic setup:
- Finite possibility space Ω
- Finite set of gambles $\mathcal{K}$ on Ω
- Lower previsions $\underline{P}$ on $\mathcal{K}$

Matrix notation:
- $|\Omega|$-by-$|\mathcal{K}|$ matrix K with gambles as columns
- the rows of K (columns of K^T) are the degenerate previsions
- the set $\mathcal{S}$ of matrices S obtained from the identity matrix $\mathbb{I}$ by changing at most one 1 to -1
- all-one (zero) column vector $\mathbb{1}$ ($\mathbb{0}$)

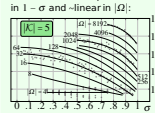
2. Goals

Given K, find a non-redundant H-representations for the set of all $\underline{P}$
A. that avoid sure loss ($[A_\mathrm{A}|\alpha_\mathrm{A}]$),
B. that avoid sure loss and for which $\underline{P} \geq \min([A_\mathrm{B}|\alpha_\mathrm{B}])$,
C. that are coherent ($[A_\mathrm{C}|\alpha_\mathrm{C}]$).

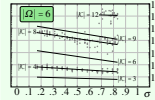
7. Experiments

The sparsity σ is the fraction of zero components in K.

Procedure C1 is exponential in $1-\sigma$ and ~linear in $|\Omega|$:



... and (at least) exponential in $|\mathcal{K}|$:



3. Goal A: Characterizing ASL

Based on the existence of a dominating linear prevision:

A1. $\exists \mu_\mathrm{L}, \nu_\mathrm{L} \geq 0$: $\underline{P} = K^\mathrm{T}\mu_\mathrm{L} - \mathbb{I}\nu_\mathrm{L} \wedge \mathbb{1}^\mathrm{T}\mu_\mathrm{L} = 1 \longrightarrow \left[\begin{smallmatrix} K^\mathrm{T} & -\mathbb{I} \\ \mathbb{1}^\mathrm{T} & \mathbb{0}^\mathrm{T} \end{smallmatrix}\right] \xrightarrow{\text{EN, RR}} [A_\mathrm{A}|\alpha_\mathrm{A}]$

A2. $\exists \mu_\mathrm{L} \geq 0$: $\underline{P} \leq K^\mathrm{T}\mu_\mathrm{L} \wedge \mathbb{1}^\mathrm{T}\mu_\mathrm{L} = 1 \longrightarrow \left[\begin{smallmatrix} \mathbb{1} & -K^\mathrm{T} & \mathbb{0} \\ & -\mathbb{I} & \mathbb{0} \\ & \mathbb{1}^\mathrm{T} & 1 \\ & -\mathbb{1}^\mathrm{T} & -1 \end{smallmatrix}\right] \xrightarrow{\text{PJ}_P\text{, RR}} [A_\mathrm{A}|\alpha_\mathrm{A}]$

4. Goal B: Characterizing ASL ≥ min

B1. Starting from $[A_\mathrm{A}|\alpha_\mathrm{A}]$: $\left[\begin{smallmatrix} A_\mathrm{A} & \alpha_\mathrm{A} \\ -\mathbb{I} & -\min \end{smallmatrix}\right] \xrightarrow{\text{RR}} [A_\mathrm{B}|\alpha_\mathrm{B}]$

5. Goal C: Characterizing coherence

Based on the existence of S-dominating linear previsions:

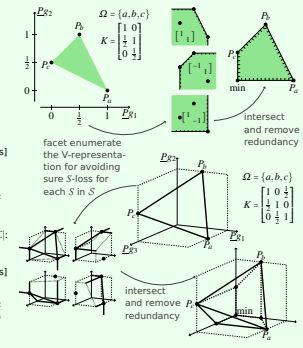
C1. Analogous to A1 & intersection over all S in $\mathcal{S}$:
$\forall S \in \mathcal{S} : \exists \mu_S, \nu_S \geq 0$: $\underline{P} = K^\mathrm{T}\mu_S - S\nu_S \wedge \mathbb{1}^\mathrm{T}\mu_S = 1 \longrightarrow \left[\begin{smallmatrix} K^\mathrm{T} & -S \\ \mathbb{1}^\mathrm{T} & \mathbb{0}^\mathrm{T} \end{smallmatrix}\right] \xrightarrow{\text{EN, IS}_{S\in\mathcal{S}}\text{, RR}} [A_\mathrm{C}|\alpha_\mathrm{C}]$

C2. Analogous to A2 & intersection over all S in $\mathcal{S}$:
$\forall S \in \mathcal{S} : \exists \mu_S \geq 0$: $\underline{P} \leq SK^\mathrm{T}\mu_S \wedge \mathbb{1}^\mathrm{T}\mu_S = 1 \longrightarrow \left[\begin{smallmatrix} S & -SK^\mathrm{T} & \mathbb{0} \\ & -\mathbb{I} & \mathbb{0} \\ & \mathbb{1}^\mathrm{T} & 1 \\ & -\mathbb{1}^\mathrm{T} & -1 \end{smallmatrix}\right] \xrightarrow{\text{PJ}_P\text{, IS}_{S\in\mathcal{S}}\text{, RR}} [A_\mathrm{C}|\alpha_\mathrm{C}]$

C3. Block matrix form of C2: $\left[\begin{smallmatrix} A_{1,\underline{P}} & A_{1,\mu_1} & & b_0 \\ \vdots & & \ddots & \vdots \\ A_{S,\underline{P}} & & A_{S,\mu_S} & b_0 \end{smallmatrix}\right] =: [A_{S,\underline{P}}\ A_{S,\mu_S}|b_0]$

$[A_{\underline{P}}\ A_\mu|b] := \left[\begin{smallmatrix} A_{1,\underline{P}} & A_{1,\mu_1} & & b_0 \\ \vdots & & \ddots & \vdots \\ A_{S,\underline{P}} & & A_{S,\mu_S} & b_0 \end{smallmatrix}\right] \xrightarrow{\text{PJ}_{\underline{P}}\text{, RR}} [A_\mathrm{C}|\alpha_\mathrm{C}]$

6. Illustrations of Procedure C1



1. Representations

Any convex polyhedron in $\mathbb{R}^n$ can be described in two ways:

H-representation (intersection of half-spaces)

$$\left[A|b\right] := \{x \in \mathbb{R}^n : Ax \leq b\}$$

(A: constraint matrix in $\mathbb{R}^{k\times n}$; b: constraint vector in $\mathbb{R}^k$)

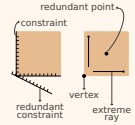
V-representation (convex hull of points and rays)

$$\left[\begin{smallmatrix} V \\ W \end{smallmatrix}\right] := \{x \in \mathbb{R}^n : x = V\mu \wedge \mu \geq 0 \wedge w^\mathrm{T}\mu = 1\}$$

(V: vector matrix in $\mathbb{R}^{n\times\ell}$; w: vector in $\mathbb{R}^\ell$; W: vector in $(\mathbb{R}^*)^{1\times\ell}$ with components defining points ($\neq 0$) and rays ($=0$))

2. Illustration

Here $n = 2$, $k = 3$, and $\ell = 4$.



3. Tasks

RR. Removing redundancy: if j is the number of non-redundant constraints (or vectors), this requires solving k (or ℓ) linear programming problems of size $n \times j$

EN. Moving between H- and V-representations: done using vertex/facet enumeration algorithms; polynomial in n, k, and ℓ.

PJ. Projection on a lower-dimensional space: easy with V-representations, hard with H-representations.

IS. Intersection: easy with H-representations, hard with V-representations.

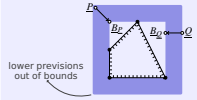
1. Context & Goal

Given: incoherent lower prevision $\underline{P}$.
Goal: Find a coherent correction to it.

2. Bring within bounds

If $\underline{P}f \in [\min f, \max f]$ for some f in $\mathcal{K}$, it is out of bounds. To bring it within bounds:

$$B_{\underline{P}}f := \begin{cases} \min f & \underline{P}f \leq \min f, \\ \max f & \underline{P}f \geq \max f, \\ \underline{P}f & \text{otherwise.} \end{cases}$$



3. Downward correction

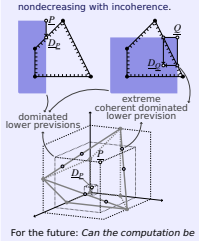
As the downward correction of $\underline{P}$ we take the lower envelope of the maximal coherent dominated lower previsions (proposed earlier by Pelessoni & Vicig, following Weichselberger), so the nadir point $D_{\underline{P}}$ of the MOLP (cf. C)

(†) maximize $\underline{Q}$, subject to $A_\mathrm{C}\underline{Q} \leq \alpha_\mathrm{C}$, $\underline{Q} \leq \underline{P}$

or the MOLP (cf. C3)

(‡) maximize $\underline{Q}$, subject to $A_{\underline{Q}}\underline{Q} + A_\mu\mu \leq b$, $\underline{Q} \leq \underline{P}$.

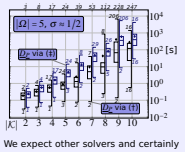
Some desirable properties:
- It is the maximal neutral correction ('no component tradeoffs').
- The imprecision of the correction is nondecreasing with incoherence.



For the future: *Can the computation be simplified for special classes of $\underline{P}$?*

4. Experiments

With the M3-solver we used, computation appears exponential in $|\mathcal{K}|$; using pre-computed constraints (†) is more efficient than not (‡):



We expect other solvers and certainly direct M2-solvers to perform more efficiently, but could not test any yet.

5. Upward correction

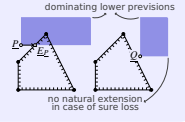
The standard upward correction of $\underline{P}$ is its natural extension $\underline{E}_{\underline{P}}$, the unique minimal pointwise dominating coherent lower prevision, so the the solution to the MOLP (cf. C)

minimize $\underline{E}_P$, subject to $A_\mathrm{C}\underline{E}_P \leq \alpha_\mathrm{C}$, $\underline{E}_P \geq \underline{P}$

or the MOLP (cf. C3)

(*) minimize $\underline{E}_P$, subject to $A_{\underline{E}_P}\underline{E}_P + A_\mu\mu \leq b$, $\underline{E}_P \geq \underline{P}$.

- The problem becomes a plain LP by using the objective $\Sigma_{g\in\mathcal{K}}\underline{E}_P g$.
- (*) decomposes into a classical formulation of natural extension.



1. Formalization

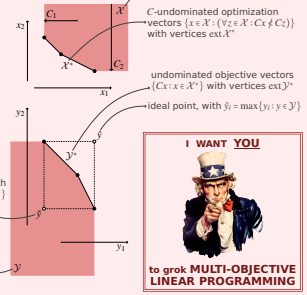
Any multi-objective linear program (MOLP) can be put in the following form:

maximize $y = Cx$, subject to $Ax \leq b$ and $x \geq 0$

(C: objective matrix in $\mathbb{R}^{m\times n}$; objective vector in $\mathbb{R}^m$; optimization vector in $\mathbb{R}^n$; constraint matrix in $\mathbb{R}^{k\times n}$; constraint vector in $\mathbb{R}^k$)

2. Illustration

Here $m = n = 2$ and $k = 4$.



3. Tasks

Main computational tasks in non-decreasing order of complexity:
M1. Finding $\tilde{y}$.
M2. Finding $\hat{y}$.
M3. Finding ext $\mathcal{Y}^*$ and characterizing $\mathcal{Y}^*$.
M4. Finding ext $\mathcal{X}^*$.
M5. Characterizing $\mathcal{X}^*$.

SYSTeMS Research Group, Ghent University — UNIVERSITEIT GENT — **Erik Quaeghebeur** — *Decision Support Systems Group, Utrecht University*

Characterizing Coherence, Correcting Incoherence

1. Context

Basic setup:
- Finite possibility space Ω
- Finite set of gambles $\mathcal{K}$ on Ω
- Lower previsions $\underline{P}$ on $\mathcal{K}$

Matrix notation:
- $|\Omega|$-by-$|\mathcal{K}|$ matrix K with gambles as columns
- the rows of K (columns of K^T) are the degenerate previsions
- the set $\mathcal{S}$ of matrices S obtained from the identity matrix $\mathbb{I}$ by changing at most one 1 to -1
- all-one (zero) column vector $\mathbb{1}$ ($\mathbb{0}$)

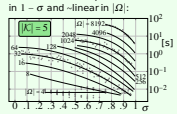
2. Goals

Given K, find a non-redundant H-representations for the set of all $\underline{P}$
A. that avoid sure loss ($[A_A|\alpha_A]$),
B. that avoid sure loss and for which $\underline{P} \geq \min([A_B|\alpha_B])$,
C. that are coherent ($[A_C|\alpha_C]$).

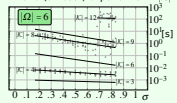
7. Experiments

The sparsity σ is the fraction of zero components in K.

Procedure C1 is exponential in $1-\sigma$ and ~linear in $|\Omega|$:



... and (at least) exponential in $|\mathcal{K}|$:



3. Goal A: Characterizing ASL

Based on the existence of a dominating linear prevision:

A1. $\exists \mu_i, \nu_i \geq 0$: $\underline{P} = K^T\mu_i - \mathbb{I}\nu_i \wedge \mathbb{1}^T\mu_i = 1 \longrightarrow \left[\begin{smallmatrix} K^T & -\mathbb{I} \\ \mathbb{1}^T & \mathbb{0}^T \end{smallmatrix}\right]$ EN, RR $\longrightarrow [A_A|\alpha_A]$

A2. $\exists \mu_i \geq 0$: $\underline{P} \leq K^T\mu_i \wedge \mathbb{1}^T\mu_i = 1 \longrightarrow \left[\begin{smallmatrix} \mathbb{I} & -K^T & \mathbb{0} \\ & -\mathbb{I} & \mathbb{0} \\ & \mathbb{1}^T & 1 \\ & -\mathbb{1}^T & -1 \end{smallmatrix}\right]$ PJ$_{\mathcal{P}}$, RR $\longrightarrow [A_A|\alpha_A]$

4. Goal B: Characterizing ASL ≥ min

B1. Starting from $[A_A|\alpha_A]$: $\left[\begin{smallmatrix} \Lambda_A & \alpha_A \\ -\mathbb{I} & -\min \end{smallmatrix}\right]$ RR $\longrightarrow [A_B|\alpha_B]$

5. Goal C: Characterizing coherence

Based on the existence of S-dominating linear previsions:

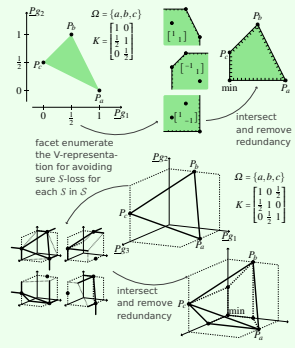
C1. Analogous to A1 & intersection over all S in $\mathcal{S}$:
$\forall S \in \mathcal{S} : \exists \mu_S, \nu_S \geq 0$: $\underline{P} = K^T\mu_S - S\nu_S \wedge \mathbb{1}^T\mu_S = 1 \longrightarrow \left[\begin{smallmatrix} K^T & -S \\ \mathbb{1}^T & \mathbb{0}^T \end{smallmatrix}\right]$ EN, IS$_{S\in\mathcal{S}}$, RR $\longrightarrow [A_C|\alpha_C]$

C2. Analogous to A2 & intersection over all S in $\mathcal{S}$:
$\forall S \in \mathcal{S} : \exists \mu_S \geq 0$: $\underline{S}\underline{P} \leq SK^T\mu_S \wedge \mathbb{1}^T\mu_S = 1 \longrightarrow \left[\begin{smallmatrix} S & -SK^T & \mathbb{0} \\ & -\mathbb{I} & \mathbb{0} \\ & \mathbb{1}^T & 1 \\ & -\mathbb{1}^T & -1 \end{smallmatrix}\right]$ PJ$_{\mathcal{P}}$, IS$_{S\in\mathcal{S}}$, RR $\longrightarrow [A_C|\alpha_C]$

C3. Block matrix form of C2: $:= [A_{S\underline{P}}\ A_{S\mu_S}|b_0]$

$\left[A_{\underline{P}}\ A_\mu|b\right] := \left[\begin{smallmatrix} A_{1\underline{P}} & A_{1\mu_1} & & \\ \vdots & & \ddots & \\ A_{S\underline{P}} & & & A_{S\mu_S} \\ \vdots & & & \end{smallmatrix}\middle|\begin{smallmatrix} b_0 \\ \vdots \\ b_0 \\ \vdots \end{smallmatrix}\right]$ PJ$_{\underline{P}}$, RR $\longrightarrow [A_C|\alpha_C]$

6. Illustrations of Procedure C1



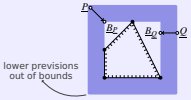
1. Context & Goal

Given: incoherent lower prevision $\underline{P}$.
Goal: Find a coherent correction to it.

2. Bring within bounds

If $\underline{P}f \in [\min f, \max f]$ for some f in $\mathcal{K}$, it is out of bounds. To bring it within bounds:

$$B_{\underline{P}}f := \begin{cases} \min f & \underline{P}f \leq \min f, \\ \max f & \underline{P}f \geq \max f, \\ \underline{P}f & \text{otherwise.} \end{cases}$$



3. Downward correction

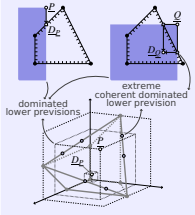
As the downward correction of $\underline{P}$ we take the lower envelope of the maximal coherent dominated lower previsions (proposed earlier by Pelessoni & Vicig, following Weichselberger), so the nadir point $D_{\underline{P}}$ of the MOLP (cf. C)

(†) maximize $\underline{Q}$, subject to $A_C\underline{Q} \leq \alpha_C$, $\underline{Q} \leq \underline{P}$

or the MOLP (cf. C3)

(‡) maximize $\underline{Q}$, subject to $A_{\underline{Q}}\underline{Q} + A_\mu\mu \leq b$, $\underline{Q} \leq \underline{P}$.

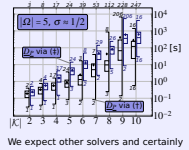
Some desirable properties:
- It is the maximal neutral correction ('no component tradeoffs').
- The imprecision of the correction is nondecreasing with incoherence.



For the future: *Can the computation be simplified for special classes of $\underline{P}$?*

4. Experiments

With the M3-solver we used, computation appears exponential in $|\mathcal{K}|$; using pre-computed constraints (†) is more efficient than not (‡):



We expect other solvers and certainly direct M2-solvers to perform more efficiently, but could not test any yet.

5. Upward correction

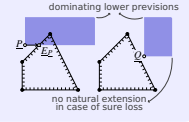
The standard upward correction of $\underline{P}$ is its natural extension $\underline{E}_{\underline{P}}$, the unique minimal pointwise dominating coherent lower prevision, so the the solution to the MOLP (cf. C)

minimize $\underline{E}_{\underline{P}}$, subject to $A_C\underline{E}_{\underline{P}} \leq \alpha_C$, $\underline{E}_{\underline{P}} \geq \underline{P}$

or the MOLP (cf. C3)

(*) minimize $\underline{E}_{\underline{P}}$, subject to $A_{\underline{E}_{\underline{P}}}\underline{E}_{\underline{P}} + A_\mu\mu \leq b$, $\underline{E}_{\underline{P}} \geq \underline{P}$.

- The problem becomes a plain LP by using the objective $\sum_{g\in\mathcal{K}} \underline{E}_{\underline{P}}g$.
- (*) decomposes into a classical formulation of natural extension.



1. Representations

Any convex polyhedron in $\mathbb{R}^n$ can be described in two ways:

H-representation (intersection of half-spaces)

$$\left[A|b\right] := \{x \in \mathbb{R}^n : Ax \leq b\}$$

(A: constraint matrix in $\mathbb{R}^{k\times n}$; b: constraint vector in $\mathbb{R}^k$)

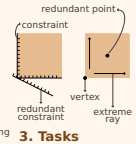
V-representation (convex hull of points and rays)

$$\left[\begin{smallmatrix} V \\ W \end{smallmatrix}\right] := \{x \in \mathbb{R}^n : x = V\mu \wedge \mu \geq 0 \wedge w^T\mu = 1\}$$

(V: vector matrix in $\mathbb{R}^{n\times\ell}$; w: vector in $\mathbb{R}^\ell$; W: vector in $(\mathbb{R}^n)_{\neq 0}$ with components defining points ($\neq 0$) and rays ($=0$))

2. Illustration

Here $n = 2$, $k = 3$, and $\ell = 4$.



3. Tasks

RR. Removing redundancy: if j is the number of non-redundant constraints (or vectors), this requires solving k (or ℓ) linear programming problems of size $n \times j$
EN. Moving between H- and V-representations: done using vertex/facet enumeration algorithms; polynomial in n, k, and ℓ.
PJ. Projection on a lower-dimensional space: easy with V-representations, hard with H-representations.
IS. Intersection: easy with H-representations, hard with V-representations.

1. Formalization

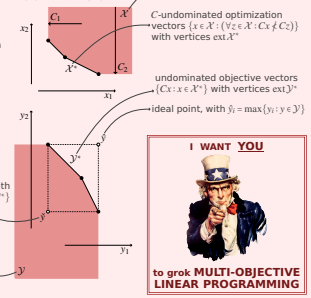
Any multi-objective linear program (MOLP) can be put in the following form:

maximize $y = Cx$, subject to $Ax \leq b$ and $x \geq 0$

(C: objective matrix in $\mathbb{R}^{m\times n}$; objective vector in $\mathbb{R}^m$; x: optimization vector in $\mathbb{R}^n$; constraint matrix in $\mathbb{R}^{k\times n}$; b: constraint vector in $\mathbb{R}^k$)

2. Illustration

Here $m = n = 2$ and $k = 4$.



3. Tasks

Main computational tasks in non-decreasing order of complexity:
M1. Finding $\hat{y}$.
M2. Finding $\check{y}$.
M3. Finding ext $\mathcal{Y}^*$ and characterizing $\mathcal{Y}^*$.
M4. Finding ext $\mathcal{X}^*$.
M5. Characterizing $\mathcal{X}^*$.

SYSTeMS Research Group, Ghent University — UNIVERSITEIT GENT — **Erik Quaeghebeur** — *Decision Support Systems Group, Utrecht University*

Polytope theory

Multi-objective linear programming

Coherence characterization procedures

Incoherence correction procedures

Polytope theory

Multi-objective linear programming

Characterizing Coherence, Correcting Incoherence

1. Context

Basic setup:
- Finite possibility space Ω
- Finite set of gambles $\mathcal{K}$ on Ω
- Lower previsions $\underline{P}$ on $\mathcal{K}$

Matrix notation:
- $|\Omega|$-by-$|\mathcal{K}|$ matrix K with gambles as columns
- the rows of K (columns of K^T) are the degenerate previsions
- the set $\mathcal{S}$ of matrices S obtained from the identity matrix $\mathbb{I}$ by changing at most one 1 to -1
- all-one (zero) column vector $\mathbb{1}$ ($\mathbb{0}$)

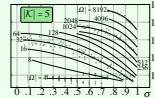
2. Goals

Given K, find a non-redundant H-representations for the set of all $\underline{P}$
A. that avoid sure loss ($[A_A|\alpha_A]$),
B. that avoid sure loss and for which $\underline{P} \geq \min([A_B|\alpha_B])$,
C. that are coherent ($[A_C|\alpha_C]$).

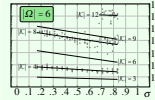
7. Experiments

The sparsity σ is the fraction of zero components in K.

Procedure C1 is exponential in $1-\sigma$ and ~linear in $|\Omega|$:



... and (at least) exponential in $|\mathcal{K}|$:



3. Goal A: Characterizing ASL

Based on the existence of a dominating linear prevision:

A1. $\exists \mu_1, \nu_1 \geq 0$: $\underline{P} = K^T\mu_1 - \mathbb{I}\nu_1 \wedge \mathbb{1}^T\mu_1 = 1 \longrightarrow \left[\begin{smallmatrix} K^T & -\mathbb{I} \\ \mathbb{1}^T & \mathbb{0}^T \end{smallmatrix}\right] \xrightarrow{\text{EN, RR}} [A_A|\alpha_A]$

A2. $\exists \mu_1 \geq 0$: $\underline{P} \leq K^T\mu_1 \wedge \mathbb{1}^T\mu_1 = 1 \longrightarrow \left[\begin{smallmatrix} \mathbb{I} & -K^T & \mathbb{0} \\ & -\mathbb{I} & \mathbb{0} \\ & \mathbb{1}^T & 1 \\ & -\mathbb{1}^T & -1 \end{smallmatrix}\right] \xrightarrow{\text{PJ}_P, \text{RR}} [A_A|\alpha_A]$

4. Goal B: Characterizing ASL ≥ min

B1. Starting from $[A_A|\alpha_A]$: $\left[\begin{smallmatrix} A_A & \alpha_A \\ -\mathbb{I} & -\min \end{smallmatrix}\right] \xrightarrow{\text{RR}} [A_B|\alpha_B]$

5. Goal C: Characterizing coherence

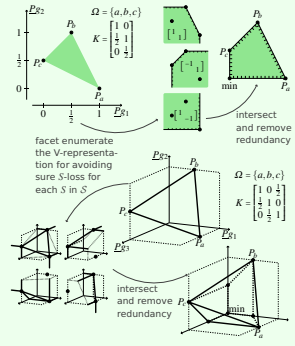
Based on the existence of S-dominating linear previsions:

C1. Analogous to A1 & intersection over all S in $\mathcal{S}$:
$\forall S \in \mathcal{S}: \exists \mu_S, \nu_S \geq 0$: $\underline{P} = K^T\mu_S - S\nu_S \wedge \mathbb{1}^T\mu_S = 1 \longrightarrow \left[\begin{smallmatrix} K^T & -S \\ \mathbb{1}^T & \mathbb{0}^T \end{smallmatrix}\right] \xrightarrow{\text{EN, IS}_{S\in\mathcal{S}}, \text{RR}} [A_C|\alpha_C]$

C2. Analogous to A2 & intersection over all S in $\mathcal{S}$:
$\forall S \in \mathcal{S}: \exists \mu_S \geq 0$: $\underline{S}\underline{P} \leq SK^T\mu_S \wedge \mathbb{1}^T\mu_S = 1 \longrightarrow \left[\begin{smallmatrix} S & -SK^T & \mathbb{0} \\ & -\mathbb{I} & \mathbb{0} \\ & \mathbb{1}^T & 1 \\ & -\mathbb{1}^T & -1 \end{smallmatrix}\right] \xrightarrow{\text{PJ}_P, \text{IS}_{S\in\mathcal{S}}, \text{RR}} [A_C|\alpha_C]$

C3. Block matrix form of C2: $[A_{\underline{P}}\ A_\mu|b] := \left[\begin{smallmatrix} A_{1,\underline{P}} & A_{1,\mu_1} & & \\ \vdots & & \ddots & \\ A_{S,\underline{P}} & & & A_{S,\mu_S} \end{smallmatrix}\middle|\begin{smallmatrix} b_0 \\ \vdots \\ b_0 \end{smallmatrix}\right] \xrightarrow{\text{PJ}_P, \text{RR}} [A_C|\alpha_C]$

6. Illustrations of Procedure C1



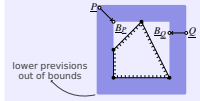
1. Context & Goal

Given: incoherent lower prevision $\underline{P}$.
Goal: Find a coherent correction to it.

2. Bring within bounds

If $\underline{P}f \notin [\min f, \max f]$ for some f in $\mathcal{K}$, it is out of bounds. To bring it within bounds:

$$\underline{B_P}f := \begin{cases} \min f & \underline{P}f \leq \min f, \\ \max f & \underline{P}f \geq \max f, \\ \underline{P}f & \text{otherwise.} \end{cases}$$



3. Downward correction

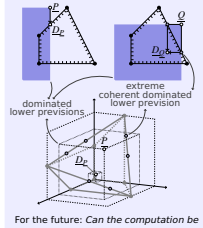
As the downward correction of $\underline{P}$ we take the lower envelope of the maximal coherent dominated lower previsions (proposed earlier by Pelessoni & Vicig, following Weichselberger), so the nadir point $\underline{D_P}$ of the MOLP (cf. C)

(†) maximize $\underline{Q}$, subject to $A_C\underline{Q} \leq \alpha_C$, $\underline{Q} \leq \underline{P}$

or the MOLP (cf. C3)

(‡) maximize $\underline{Q}$, subject to $A_{\underline{Q}}\underline{Q} + A_\mu\mu \leq b$, $\underline{Q} \leq \underline{P}$.

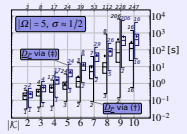
Some desirable properties:
- It is the maximal neutral correction ('no component tradeoffs').
- The imprecision of the correction is nondecreasing with incoherence.



For the future: *Can the computation be simplified for special classes of $\underline{P}$?*

4. Experiments

With the M3-solver we used, computation appears exponential in $|\mathcal{K}|$; using pre-computed constraints (†) is more efficient than not (‡):



We expect other solvers and certainly direct M2-solvers to perform more efficiently, but could not test any yet.

5. Upward correction

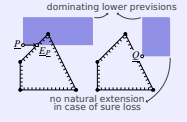
The standard upward correction of $\underline{P}$ is its natural extension $\underline{E_P}$, the unique minimal pointwise dominating coherent lower prevision, so the the solution to the MOLP (cf. C)

minimize $\underline{E_P}$, subject to $A_C\underline{E_P} \leq \alpha_C$, $\underline{E_P} \geq \underline{P}$

or the MOLP (cf. C3)

(*) minimize $\underline{E_P}$, subject to $A_{\underline{E_P}}\underline{E_P} + A_\mu\mu \leq b$, $\underline{E_P} \geq \underline{P}$.

- The problem becomes a plain LP by using the objective $\sum_{g\in\mathcal{K}} \underline{E_P}g$.
- (*) decomposes into a classical formulation of natural extension.



1. Representations

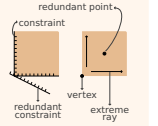
Any convex polyhedron in $\mathbb{R}^n$ can be described in two ways:

H-representation (intersection of half-spaces)
$$\left[A\middle|b\right] := \{x \in \mathbb{R}^n : Ax \leq b\}$$
(constraint matrix in $\mathbb{R}^{k\times n}$, constraint vector in $\mathbb{R}^k$)

V-representation (convex hull of points and rays)
$$\left[\begin{smallmatrix} V \\ W \end{smallmatrix}\right] := \{x \in \mathbb{R}^n : x = V\mu \wedge \mu \geq 0 \wedge w^T\mu = 1\}$$
(vector matrix in $\mathbb{R}^{n\times\ell}$, vector in $\mathbb{R}^\ell$, vector in $(\mathbb{R}^\ell)_{\geq 0}$ with components defining points ($\neq 0$) and rays ($=0$))

2. Illustration

Here $n = 2$, $k = 3$, and $\ell = 4$.



3. Tasks

RR. Removing redundancy: if j is the number of non-redundant constraints (or vectors), this requires solving k (or ℓ) linear programming problems of size $n \times j$
EN. Moving between H- and V-representations: done using vertex/facet enumeration algorithms; polynomial in n, k, and ℓ.
PJ. Projection on a lower-dimensional space: easy with V-representations, hard with H-representations.
IS. Intersection: easy with H-representations, hard with V-representations.

1. Formalization

Any multi-objective linear program (MOLP) can be put in the following form:

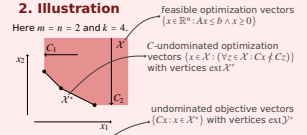
objective matrix in $\mathbb{R}^{m\times n}$, objective vector in $\mathbb{R}^m$, optimization vector in $\mathbb{R}^n$

maximize $y = Cx$, subject to $Ax \leq b$ and $x \geq 0$

constraint matrix in $\mathbb{R}^{k\times n}$, constraint vector in $\mathbb{R}^k$

2. Illustration

Here $m = n = 2$ and $k = 4$.



3. Tasks

Main computational tasks in non-decreasing order of complexity:
M1. Finding $\hat{y}$.
M2. Finding $\check{y}$.
M3. Finding ext $\mathcal{Y}^*$ and characterizing $\mathcal{Y}^*$.
M4. Finding ext $\mathcal{X}^*$.
M5. Characterizing $\mathcal{X}^*$.

nadir point, with $\check{y}_i = \min\{y_i : y \in \mathcal{Y}^*\}$

feasible objective vectors $\{Cx : x \in \mathcal{X}\}$

SYSTeMS Research Group *Ghent University* UNIVERSITEIT GENT **Erik Quaeghebeur** *Decision Support Systems Group* *Utrecht University*

I WANT YOU

to COME TO MY POSTER