Modeling uncertainty using accept & reject statements

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The setup

- Experiment with outcomes in some possibility space $\Omega$.
- Agent uncertain about the experiment’s outcome.
- Linear space $\mathcal{L}$ of real-valued gambles on $\Omega$.

Agent expresses uncertainty by making statements about gambles, forming an assessment.

Agent wishes to rationally deduce inferences and draw conclusions from this assessment.
The work we build on

- De Finetti: previsions $P$.

\[ f \Rightarrow \text{sure loss} \]

\[ Pg = 0 \]

\[ f - Pf \]

\[ f - Pf \Rightarrow \text{set of desirable gambles} \]
The work we build on

- De Finetti: previsions $P$.

- Williams, Seidenfeld et al., Walley:
  - lower previsions $P$,
  - sets of acceptable/favorable/desirable gambles,
  - partial preference orders $\succeq$.
Accepting & Rejecting Gambles

**Accepting** a gamble $f$ implies a commitment to engage in the following transaction:

(i) the experiment’s outcome $\omega \in \Omega$ is determined,

(ii) the agent gets the—possibly negative—payoff $f(\omega)$.

**Rejecting** a gamble: the agent considers accepting it unreasonable.
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**Rejecting** a gamble: the agent considers accepting it unreasonable.

**Assessment** A pair $\mathcal{A} := \langle \mathcal{A}_\geq; \mathcal{A}_\leq \rangle$ of sets of accepted and rejected gambles.
Gamble Categorization

- **Accepted** \( \mathcal{A}_\geq \).  
- **Rejected** \( \mathcal{A}_< \).  
- **Unresolved** Neither accepted nor rejected; \( \mathcal{A}_\ominus := \mathcal{L} \setminus (\mathcal{A}_\geq \cup \mathcal{A}_<) \).  
- **Confusing** Both accepted and rejected; \( \mathcal{A}_\oplus := \mathcal{A}_\geq \cap \mathcal{A}_< \).

\[
\begin{array}{c}
\oplus \\
\downarrow \\
\nearrow \\
\mathcal{A}_\geq \cap \mathcal{A}_<
\end{array}
\]
Gamble Categorization

Accepted  $\mathcal{A}_\geq$.

Rejected  $\mathcal{A}_\leq$.

Unresolved Neither accepted nor rejected; $\mathcal{A}_\ominus := \mathcal{L} \setminus (\mathcal{A}_\geq \cup \mathcal{A}_\leq)$.

Confusing Both accepted and rejected; $\mathcal{A}_\equiv := \mathcal{A}_\geq \cap \mathcal{A}_\leq$.

Indifferent Both it and its negation accepted; $\mathcal{A}_\ominus := \mathcal{A}_\geq \cap \neg \mathcal{A}_\geq$.

Favorable Accepted with a rejected negation; $\mathcal{A}_\succ := \mathcal{A}_\geq \cap \neg \mathcal{A}_\leq$.

Incomparable Both it and its negation unresolved; $\mathcal{A}_\vartriangledown := \mathcal{A}_\ominus \cap \neg \mathcal{A}_\ominus$. 
The first rationality axiom: **No Confusion**

Because of the interpretation attached to acceptance and rejection statements, we consider confusion irrational.

So we require assessments $\mathcal{A}$ to not contain confusion:

$$\mathcal{A}_\emptyset = \mathcal{A}_\geq \cap \mathcal{A}_\leq = \emptyset$$
Deductive extension

We assume gamble payoffs are expressed in a linear precise utility scale, so:

- combinations of accepted gambles are acceptable ($\mathcal{K} + \mathcal{K}$).
- positively scaled accepted gambles are acceptable ($\overline{\mathcal{K}}$).

The positive linear hull operator $\text{posi}$ combines both operations; it generates convex cones.
Deductive extension

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- positively scaled accepted gambles are acceptable ($\overline{\mathcal{K}}$).

The positive linear hull operator $\text{posi}$ combines both operations; it generates convex cones.

An assessment $\mathcal{A}$ can be deductively extended to a deductively closed assessment $\mathcal{D} := \langle \text{posi} \mathcal{A}_\geq; \mathcal{A}_\leq \rangle$. 
The second rationality axiom: **Deductive Closure**

The assumption of a linear precise utility scale leads us to exclusively use deductively closed assessments $\mathcal{D}$ for inference and decision purposes:

$$\text{posi } \mathcal{D}_\geq = \mathcal{D}_\geq$$
Gambles in limbo & reckoning extension

Deductive Closure interacts with No Confusion:

- Consider a deductively closed assessment $\mathcal{D}$.
- Additionally consider some unresolved gamble $f$ acceptable.
- Apply deductive extension to $\langle \mathcal{D}_\geq \cup \{f\}; \mathcal{D}_< \rangle$.
- For some $f$, this would lead to an increase in confusion.
- These have the same effect as gambles in $\mathcal{D}_<$, and form the limbo $(\overline{\mathcal{D}_< \setminus \mathcal{D}_\geq} - (\mathcal{D}_\geq \cup \{0\})) \setminus \mathcal{D}_<$ of $\mathcal{D}$. 
Gambles in limbo & reckoning extension

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- Consider a deductively closed assessment $\mathcal{D}$.
- Additionally consider some unresolved gamble $f$ acceptable.
- Apply deductive extension to $\langle \mathcal{D}_\geq \cup \{f\}; \mathcal{D}_< \rangle$.
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We use reckoning extension to reject gambles in limbo and create a model $\mathcal{M} := \langle \mathcal{D}_\geq; \mathcal{D}_\leq \cup (\overline{\mathcal{D}_\leq \setminus \mathcal{D}_\leq} - (\mathcal{D}_\geq \cup \{0\})) \rangle$. 
The third rationality axiom: **No Limbo**

We consider ignoring gambles in limbo unreasonable and therefore further restrict attention to *models* $\mathcal{M}$ for inference and decision purposes:

$$
(\overline{\mathcal{M}_< \setminus \mathcal{M}_\ge}) - (\mathcal{M}_\ge \cup \{0\}) \subseteq \mathcal{M}_<
$$
The fourth rationality axiom: **Indifference to Status Quo**

Because there is no adverse effect, it is not unreasonable to accept the zero gamble $0$, status quo; because it is convenient, we find it reasonable:

$$0 \in \mathcal{M}_\geq$$
Main characterization result

An assessment $\mathcal{M}$ is a model that satisfies No Confusion and Indifference to Status Quo iff

(i) $0 \in \mathcal{M}_\geq$,  
(ii) $0 \notin \mathcal{M}_<$,  
(iii) $\text{posi } \mathcal{M}_\geq = \mathcal{M}_\geq$,  
(iv) $\mathcal{M}_< - \mathcal{M}_\geq \subseteq \mathcal{M}_<$. 
Main characterization result

An assessment $\mathcal{M}$ is a model that satisfies No Confusion and Indifference to Status Quo iff

(i) $0 \in \mathcal{M}_\geq$,
(ii) $0 \notin \mathcal{M}_<$,
(iii) $\text{posi } \mathcal{M}_\geq = \mathcal{M}_\geq$,
(iv) $\mathcal{M}_< - \mathcal{M}_\geq \subseteq \mathcal{M}_<$. 

These partition gamble space as follows:
Gamble relations

- \( f \) is accepted in exchange for \( h \): \( f \geq h \iff f - h \in \mathcal{M}_\geq \).

\[
f - 2 \cdot g \quad 2 \cdot g - f
\]
Gamble relations

- $f$ is accepted in exchange for $h$: $f \geq h \iff f - h \in \mathcal{M}_{\geq}$.
- $f$ is dispreferred to $h$: $f < h \iff f - h \in \mathcal{M}_{<}$. 
Gamble relations

- $f$ is accepted in exchange for $h$: $f \succeq h \iff f - h \in \mathcal{M}_\succeq$.
- $f$ is dispreferred to $h$: $f < h \iff f - h \in \mathcal{M}_<$.

- Indifference between $f$ and $h$: $f \simeq h \iff f \geq h \land h \geq f \iff f - h \in \mathcal{M}_\simeq$. 

The diagram illustrates the relationships between $f$, $g$, $f'$, and $g'$, with $f$ being preferred over $h$, indifference between $f$ and $h$, and $f'$ and $g'$ being incomparable.
Gamble relations

- $f$ is accepted in exchange for $h$: $f \succeq h \iff f - h \in \mathcal{M}_\geq$.
- $f$ is dispreferred to $h$: $f \prec h \iff f - h \in \mathcal{M}_<$.

- Indifference between $f$ and $h$: $f \asymp h \iff f \geq h \land h \geq f \iff f - h \in \mathcal{M}_\sim$.
- $f$ is preferred over $h$: $f \succ h \iff f \geq h \land h \prec f \iff f - h \in \mathcal{M}_\succ$. 

![Diagram showing the relation between different gambles and the differences in their values.](image)
Gamble relations

- $f$ is accepted in exchange for $h$: $f \geq h \iff f - h \in \mathcal{M}_\geq$.
- $f$ is dispreferred to $h$: $f < h \iff f - h \in \mathcal{M}_<$.

- Indifference between $f$ and $h$: $f \sim h \iff f \geq h \land h \geq f \iff f - h \in \mathcal{M}_\sim$.
- $f$ is preferred over $h$: $f \succ h \iff f \geq h \land h < f \iff f - h \in \mathcal{M}_\succ$.
- $f$ and $h$ are incomparable: $f \asymp h \iff f - h \in \mathcal{M}_\asymp$. 
Gamble relations $\geq$ and $<$ are equivalent to a model that satisfies No Confusion and Indifference to Status Quo iff

(i) Accept Reflexivity: $f \geq f$,

(ii) Reject Irreflexivity: $f \nless f$,

(iii) Accept Transitivity: $f \geq g \land g \geq h \Rightarrow f \geq h$.

(iv) Mixed Transitivity: $f \less g \land h \geq g \Rightarrow f \less h$,

(v) Mixture independence: $f \geq g \iff \mu \cdot f + (1-\mu) \cdot h \geq \mu \cdot g + (1-\mu) \cdot h$. 

Acceptability $\geq$ is a non-strict pre-order (a vector ordering).

Indifference $\equiv$ is an equivalence relation.

Preference $\prec$ is a strict partial order.
Characterization result for gamble relations

Gamble relations $\succeq$ and $<$ are equivalent to a model that satisfies No Confusion and Indifference to Status Quo iff

(i) Accept Reflexivity: $f \succeq f$,
(ii) Reject Irreflexivity: $f \not< f$,
(iii) Accept Transitivity: $f \geq g \land g \geq h \Rightarrow f \geq h$.
(iv) Mixed Transitivity: $f < g \land h \geq g \Rightarrow f < h$,
(v) Mixture independence: $f \geq g \iff \mu \cdot f + (1 - \mu) \cdot h \geq \mu \cdot g + (1 - \mu) \cdot h$.

- Acceptability $\succeq$ is a non-strict pre-order (a vector ordering).
- Indifference $\simeq$ is an equivalence relation.
- Preference $\succ$ is a strict partial order.
Conclusions

Our framework

- generalizes existing linear precise utility based generalizations of probability theory,

- elegantly combines distinct strict and non-strict preference orders,

- flexible on input (assessment/elicitation) and output (inference/decisions) side,

- puts the appealing ‘sets of gambles’-based approaches in the spotlight.
Want to know more: read the full paper!

http://arxiv.org/abs/1208.4462