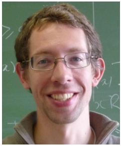
# Modeling uncertainty using accept & reject statements

Erik Quaeghebeur, Gert de Cooman, Filip Hermans



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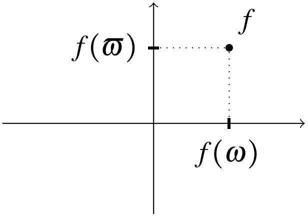
# The setup

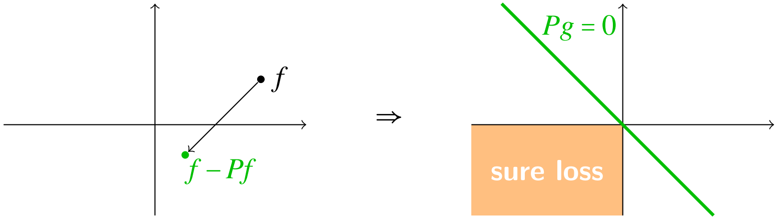
- Experiment with outcomes in some possibility space $\Omega$.
- Agent uncertain about the experiment's outcome.
- Linear space $\mathcal{L}$ of real-valued gambles on $\Omega$.




- Agent expresses uncertainty by making statements about gambles, forming an assessment.
- Agent wishes to rationally deduce inferences and draw conclusions from this assessment.

# The work we build on

- De Finetti: previsions $P$.



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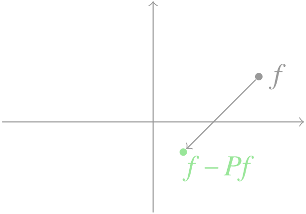
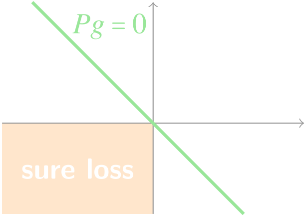
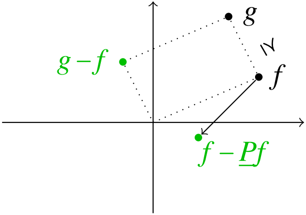



$\Rightarrow$




- Williams, Seidenfeld et al., Walley:
  - lower previsions $\underline{P}$,
  - sets of acceptable/favorable/desirable gambles,
  - partial preference orders $\geq$.

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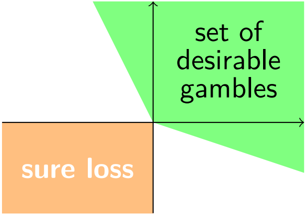

# Accepting & Rejecting Gambles

**Accepting** a gamble $f$ implies a commitment to engage in the following transaction:

(i) the experiment's outcome $\omega \in \Omega$ is determined,

(ii) the agent gets the—possibly negative—payoff $f(\omega)$.

**Rejecting** a gamble: the agent considers accepting it unreasonable.

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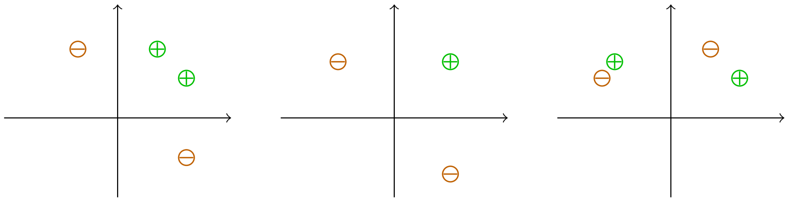
Assessment A pair $\mathcal{A} \coloneqq \langle \mathcal{A}_{\geq}; \mathcal{A}_{\prec} \rangle$ of sets of accepted and rejected gambles.



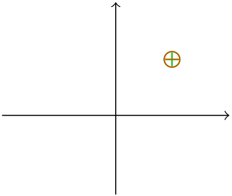
# Gamble Categorization

**Accepted** $\mathcal{A}_{\geq}$.

**Rejected** $\mathcal{A}_{\prec}$.

**Unresolved** Neither accepted nor rejected; $\mathcal{A}_{\smile} \coloneqq \mathcal{L} \setminus (\mathcal{A}_{\geq} \cup \mathcal{A}_{\prec})$.

**Confusing** Both accepted and rejected; $\mathcal{A}_{\between} \coloneqq \mathcal{A}_{\geq} \cap \mathcal{A}_{\prec}$.



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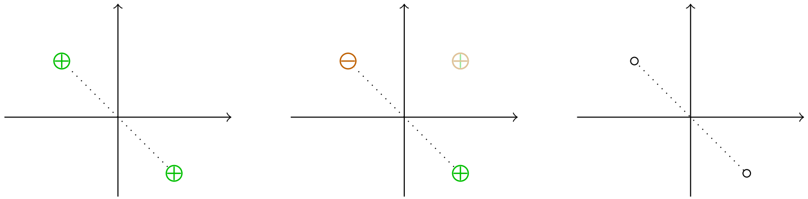

**Indifferent** Both it and its negation accepted; $\mathcal{A}_{\simeq} \coloneqq \mathcal{A}_{\geq} \cap -\mathcal{A}_{\geq}$.

**Favorable** Accepted with a rejected negation; $\mathcal{A}_{\succ} \coloneqq \mathcal{A}_{\geq} \cap -\mathcal{A}_{\prec}$.

**Incomparable** Both it and its negation unresolved; $\mathcal{A}_{\asymp} \coloneqq \mathcal{A}_{\smile} \cap -\mathcal{A}_{\smile}$.

# The first rationality axiom: **No Confusion**

Because of the interpretation attached to acceptance and rejection statements, we consider confusion irrational.

So we require assessments $\mathcal{A}$ to not contain confusion:

$$\mathcal{A}_{\between} = \mathcal{A}_{\succeq} \cap \mathcal{A}_{\prec} = \emptyset$$

# Deductive extension

We assume gamble payoffs are expressed in a linear precise utility scale, so:

- combinations of accepted gambles are acceptable ($\mathcal{K}+\mathcal{K}$).
- positively scaled accepted gambles are acceptable ($\overline{\mathcal{K}}$).

The positive linear hull operator $\operatorname{posi}$ combines both operations;
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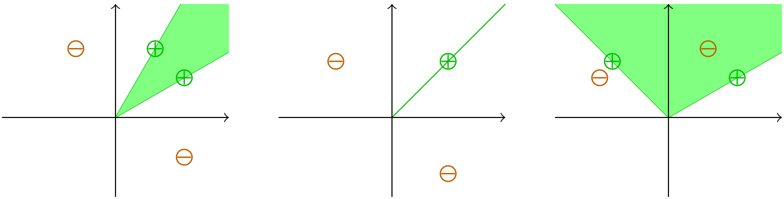
An assessment $\mathcal{A}$ can be deductively extended to
a deductively closed assessment $\mathcal{D} \coloneqq \langle \operatorname{posi} \mathcal{A}_{\geq} ; \mathcal{A}_{<} \rangle$.



# The second rationality axiom: **Deductive Closure**

The assumption of a linear precise utility scale leads us to exclusively use deductively closed assessments $\mathcal{D}$ for inference and decision purposes:

$$\operatorname{posi} \mathcal{D}_{\geq} = \mathcal{D}_{\geq}$$

# Gambles in limbo & reckoning extension

Deductive Closure interacts with No Confusion:

- Consider a deductively closed assessment $\mathcal{D}$.
- Additionally consider some unresolved gamble $f$ acceptable.
- Apply deductive extension to $\langle \mathcal{D}_{\geq} \cup \{f\}; \mathcal{D}_{\prec} \rangle$.
- For some $f$, this would lead to an increase in confusion.
- These have the same effect as gambles in $\mathcal{D}_{\prec}$,
  and form the limbo $\big((\overline{\mathcal{D}_{\prec}} \setminus \mathcal{D}_{\geq}) - (\mathcal{D}_{\geq} \cup \{0\})\big) \setminus \mathcal{D}_{\prec}$ of $\mathcal{D}$.

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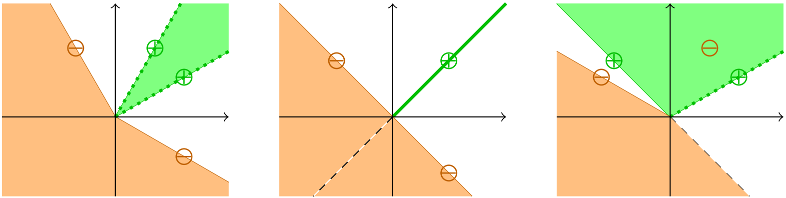
We use reckoning extension to reject gambles in limbo
and create a model $\mathcal{M} \coloneqq \Big\langle \mathcal{D}_{\succeq}; \mathcal{D}_{\prec} \cup \Big((\overline{\mathcal{D}_{\prec}} \setminus \mathcal{D}_{\succeq}) - (\mathcal{D}_{\succeq} \cup \{0\})\Big)\Big\rangle$.



# The third rationality axiom: **No Limbo**

We consider ignoring gambles in limbo unreasonable and therefore further restrict attention to models $\mathcal{M}$ for inference and decision purposes:

$$(\overline{\mathcal{M}_{\prec}} \setminus \mathcal{M}_{\geq}) - (\mathcal{M}_{\geq} \cup \{0\}) \subseteq \mathcal{M}_{\prec}$$

# The fourth rationality axiom: **Indifference to Status Quo**

Because there is no adverse effect, it is not unreasonable to accept the zero gamble $0$, status quo; because it is convenient, we find it reasonable:

$$0 \in \mathcal{M}_{\geq}$$

# Main characterization result

An assessment $\mathcal{M}$ is a model that satisfies
No Confusion and Indifference to Status Quo iff

(i) $0 \in \mathcal{M}_{\succeq}$,

(ii) $0 \notin \mathcal{M}_{\prec}$,

(iii) $\operatorname{posi} \mathcal{M}_{\succeq} = \mathcal{M}_{\succeq}$,

(iv) $\mathcal{M}_{\prec} - \mathcal{M}_{\succeq} \subseteq \mathcal{M}_{\prec}$.

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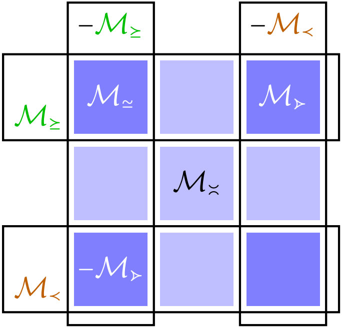
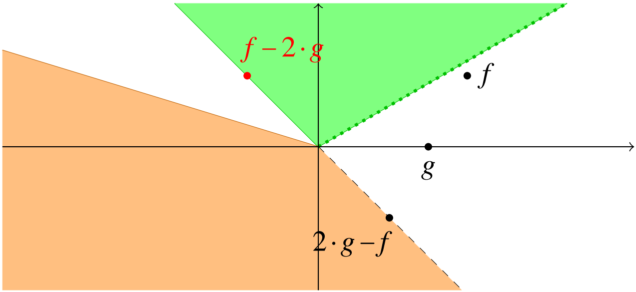
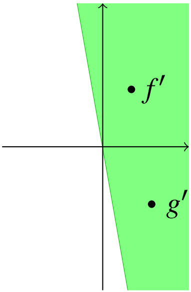
These partition gamble space as follows:



# Gamble relations

- $f$ is accepted in exchange for $h$: $f \geq h \Leftrightarrow f - h \in \mathcal{M}_{\geq}$.



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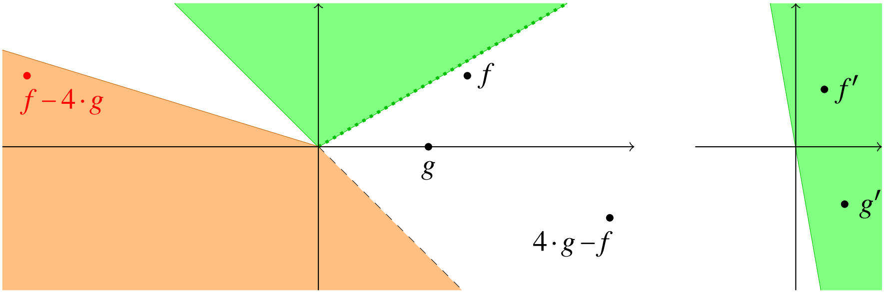
- $f$ is dispreferred to $h$: $f \prec h \Leftrightarrow f - h \in \mathcal{M}_{\prec}$.



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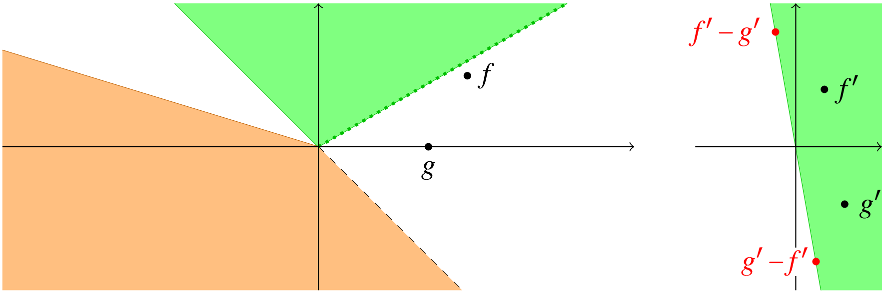
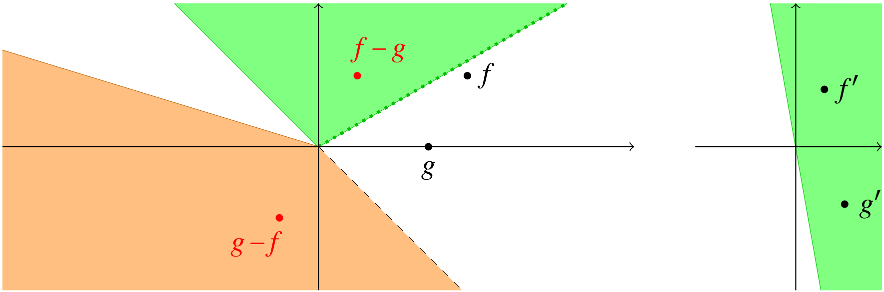



- indifference between $f$ and $h$: $f \simeq h \Leftrightarrow f \geq h \wedge h \geq f \Leftrightarrow f - h \in \mathcal{M}_{\simeq}$.

# Gamble relations

- $f$ is accepted in exchange for $h$: $f \succeq h \Leftrightarrow f - h \in \mathcal{M}_{\succeq}$.
- $f$ is dispreferred to $h$: $f \prec h \Leftrightarrow f - h \in \mathcal{M}_{\prec}$.




- indifference between $f$ and $h$: $f \simeq h \Leftrightarrow f \succeq h \wedge h \succeq f \Leftrightarrow f - h \in \mathcal{M}_{\simeq}$.
- $f$ is preferred over $h$: $f \succ h \Leftrightarrow f \succeq h \wedge h \prec f \Leftrightarrow f - h \in \mathcal{M}_{\succ}$.

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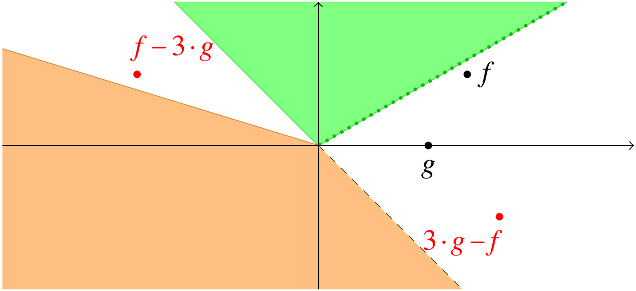


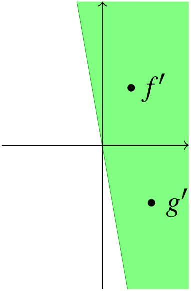


- indifference between $f$ and $h$: $f \simeq h \Leftrightarrow f \geq h \wedge h \geq f \Leftrightarrow f - h \in \mathcal{M}_{\simeq}$.
- $f$ is preferred over $h$: $f \succ h \Leftrightarrow f \geq h \wedge h \prec f \Leftrightarrow f - h \in \mathcal{M}_{\succ}$.
- $f$ and $h$ are incomparable: $f \asymp h \Leftrightarrow f - h \in \mathcal{M}_{\asymp}$.

# Characterization result for gamble relations

Gamble relations $\succeq$ and $\prec$ are equivalent to a model that satisfies No Confusion and Indifference to Status Quo iff

(i) Accept Reflexivity: $f \succeq f$,

(ii) Reject Irreflexivity: $f \not\prec f$,

(iii) Accept Transitivity: $f \succeq g \wedge g \succeq h \Rightarrow f \succeq h$.

(iv) Mixed Transitivity: $f \prec g \wedge h \succeq g \Rightarrow f \prec h$,

(v) Mixture independence: $f \succeq g \Leftrightarrow \mu \cdot f + (1-\mu) \cdot h \succeq \mu \cdot g + (1-\mu) \cdot h$.

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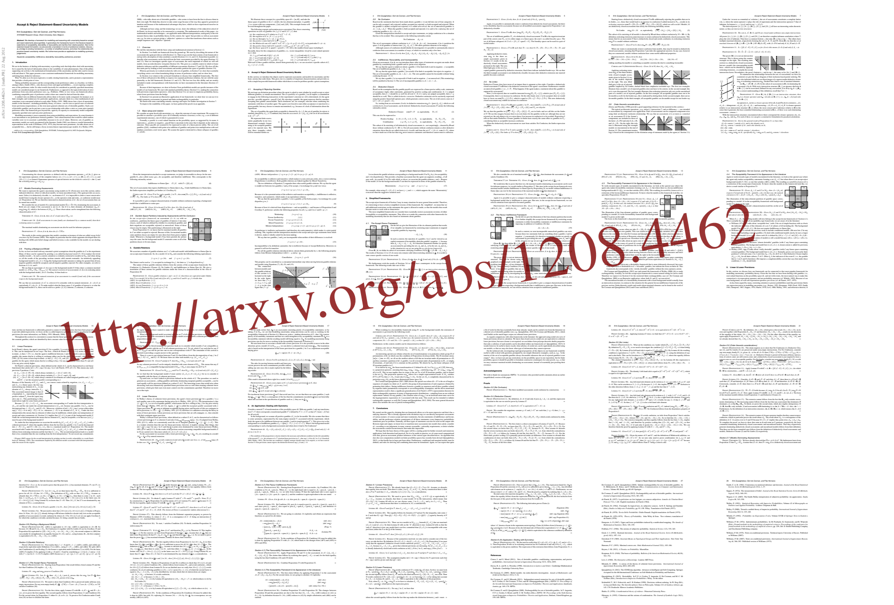
- Acceptability $\geq$ is a non-strict pre-order (a vector ordering).
- Indifference $\simeq$ is an equivalence relation.
- Preference $\succ$ is a strict partial order.

# Conclusions

Our framework

- generalizes existing linear precise utility based generalizations of probability theory,
- elegantly combines distinct strict and non-strict preference orders,
- flexible on input (assessment/elicitation) and output (inference/decisions) side,
- puts the appealing 'sets of gambles'-based approaches in the spotlight.

# Want to know more: read the full paper!



http://arxiv.org/abs/1208.4462