MODELING UNCERTAINTY USING ACCEPT & REJECT STATEMENTS

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Uncertainty and preference is often modeled using linear previsions and linear orders. Some more expressive models use sets of probabilities, lower previsions, or partial orders (see, e.g., Seidenfeld et al., 1990; Walley, 1991). In the discussion of these more expressive models, or even to justify them, alternative representations in terms of sets of so-called acceptable, favorable, or desirable gambles appear (cf. Williams, 1974, 1975; Seidenfeld et al., 1990; Walley, 1991, 2000). Such 'sets of gambles'-based models are attractive because of their geometric nature.

We generalize these 'sets of gambles'-based models by considering a *pair* of sets, one with accepted gambles and one with rejected gambles. We develop a framework based on a small number of axioms and provide an interesting characterization of the resulting models. Furthermore, we define a pair of equivalent gamble relations that generalize the partial orders mentioned earlier; the corresponding characterization result is also given.

Accepting & Rejecting Gambles Consider an agent faced with uncertainty, formalized using a linear space $\mathcal{L}$ of gambles with payoffs expressed in units of a linear precise utility. We envisage an elicitation procedure where the agent is asked to state whether he would *accept* a gamble—and its possibly negative outcome—, *reject* it—if he considers it unreasonable to accept—, or remain uncommitted.

The agent's set of acceptable gambles is $\mathcal{A}_{\geq}$; the set of gambles he rejects is $\mathcal{A}_{<}$. They form his *assessment* $\mathcal{A} := \langle \mathcal{A}_{\geq}; \mathcal{A}_{<} \rangle$; the set of all assessments is $\mathbf{A} := 2^{\mathcal{L}} \times 2^{\mathcal{L}}$.

In terms of statements, a gamble f can fall into one of four categories: only accepted, only rejected, *unresolved*—neither accepted nor rejected; $\mathcal{A}_{\smile} := \mathcal{L} \setminus (\mathcal{A}_{\geq} \cup \mathcal{A}_{<})$—, or *confusing*—both accepted and rejected.

Derived statements Given an assessment $\mathcal{A}$ in $\mathbf{A}$, we can introduce three other types of statements by considering both a gamble and its (pointwise) negation.

The agent is *indifferent* about a gamble f if he finds both it and its negation $-f$ acceptable; $\mathcal{A}_{\simeq} := \mathcal{A}_{\geq} \cap -\mathcal{A}_{\geq}$.[1]

The agent finds a gamble f *favorable* if he finds it acceptable, but rejects its negation $-f$; $\mathcal{A}_{\triangleright} := \mathcal{A}_{\geq} \cap -\mathcal{A}_{<}$. A gamble f is *incomparable* if both it and its negation $-f$ are unresolved; $\mathcal{A}_{\asymp} := \mathcal{A}_{\smile} \cap -\mathcal{A}_{\smile}$.

No Confusion We judge confusion to be a situation that has to be avoided. This corresponds to the following axiom:

$$\text{No Confusion:} \quad \mathcal{A}_{\geq} \cap \mathcal{A}_{<} = \varnothing. \tag{1}$$

The set of assessments without confusion is denoted by $\mathbb{A}$.

Deductive Closure Based on the assumption that the gamble payoffs are expressed in a linear precise utility scale, statements of acceptance imply other statements, generated by positive scaling and addition. This is called *deductive extension*. Starting from an assessment $\mathcal{A}$ in $\mathbf{A}$, its deductive extension is $\langle \operatorname{posi} \mathcal{A}_{\geq}; \mathcal{A}_{<} \rangle$.[2]

Deductively closed assessments $\mathcal{D}$ satisfy the following axiom:

$$\text{Deductive Closure:} \quad \operatorname{posi} \mathcal{D}_{\geq} = \mathcal{D}_{\geq}. \tag{2}$$

The subset of $\mathbf{A}$ consisting of all deductively closed assessments is denoted by $\mathbf{D}$ and those without confusion by $\mathbb{D} := \mathbf{D} \cap \mathbb{A}$.

No Limbo Deductive Closure does have more of an impact than is apparent at first sight. Consider a deductively closed assessment $\mathcal{D}$ in $\mathbf{D}$ that is the deductive extension of the agent's assessment. It can be shown that under Deductive Closure, the gambles in $(\overline{\mathcal{D}_{<}} \setminus \mathcal{D}_{\geq}) - (\mathcal{D}_{\geq} \cup \{0\})$ have exactly the same effect as gambles in $\mathcal{D}_{<}$:[3] considering them acceptable increases confusion. We call $\big((\overline{\mathcal{D}_{<}} \setminus \mathcal{D}_{\geq}) - (\mathcal{D}_{\geq} \cup \{0\})\big) \setminus \mathcal{D}_{<}$ the *limbo* of $\mathcal{D}$.

Starting from a deductively closed assessment $\mathcal{D}$ in $\mathbf{D}$, additionally rejecting the gambles that are in its limbo results in its *reckoning extension*

$$\Big\langle \mathcal{D}_{\geq}; \mathcal{D}_{<} \cup \big((\overline{\mathcal{D}_{<}} \setminus \mathcal{D}_{\geq}) - (\mathcal{D}_{\geq} \cup \{0\})\big) \Big\rangle,$$

[1] $-\mathcal{K} := \{-g : g \in \mathcal{K}\}$.

[2] $\operatorname{posi} \mathcal{K} := \bigcup \big\{ \sum_{g \in \mathcal{K}'} \lambda_g \cdot g : \mathcal{K}' \subseteq \overline{\mathcal{K}} \wedge |\mathcal{K}'| \in \mathbb{N} \wedge \lambda \in \mathbb{R}_{>0}^{\mathcal{K}'} \big\}$.

[3] $\overline{\mathcal{K}} := \bigcup_{f \in \mathcal{K}} \{\lambda f : \lambda \in \mathbb{R}_{>0}\}$ and $\mathcal{K} + \mathcal{K}' := \{g + h : g \in \mathcal{K} \wedge h \in \mathcal{K}'\}$.

which we call a *model*. So models $\mathcal{M}$ are deductively closed assessments that satisfy the following axiom:

$$\text{No Limbo:} \quad (\overline{\mathcal{M}_{\prec}} \setminus \mathcal{M}_{\succeq}) - (\mathcal{M}_{\succeq} \cup \{0\}) \subseteq \mathcal{M}_{\prec}. \quad (3)$$

The subset of $\mathbf{A}$ consisting of all models is denoted by $\mathbf{M}$ and those without confusion by $\mathbb{M} := \mathbf{M} \cap \mathbb{A}$.

Indifference to Status Quo We judge it reasonable to always let the zero gamble 0—also called *status quo*—be acceptable, and therefore indifferent. This corresponds to the following axiom:

$$\text{Indifference to Status Quo:} \quad 0 \in \mathcal{A}_{\succeq}. \quad (4)$$

The set of assessments satisfying Indifference to Status Quo is denoted by $\mathbf{A}_0$ (similarly, $\mathbb{M}_0$ etc.).

Main characterization result Given $\mathcal{M}$ in $\mathbf{A}$, then $\mathcal{M} \in \mathbb{M}_0$ if and only if
(i) $0 \in \mathcal{M}_{\succeq}$,
(ii) $0 \notin \mathcal{M}_{\prec}$,
(iii) $\operatorname{posi} \mathcal{M}_{\succeq} = \mathcal{M}_{\succeq}$,
(iv) $\mathcal{M}_{\prec} - \mathcal{M}_{\succeq} \subseteq \mathcal{M}_{\prec}$.

Gamble Relations We associate a pair of gamble relations on $\mathcal{L} \times \mathcal{L}$ with each model $\mathcal{M}$ in $\mathbb{M}_{\mathcal{O}}$:

$$f \succeq g \Leftrightarrow f - g \in \mathcal{M}_{\succeq} \quad \text{and} \quad f \prec g \Leftrightarrow f - g \in \mathcal{M}_{\prec}. \quad (5)$$

The former can be read as 'f is accepted in exchange for g', the latter as 'f is dispreferred to g'.

The two definitions of Equation (5) engender three other useful gamble relations: The agent is *indifferent* between two gambles f and g if he accepts f in exchange for g and vice versa: $f \simeq g \Leftrightarrow f \succeq g \wedge g \succeq f \Leftrightarrow f - g \in \mathcal{M}_{\simeq}$. The agent prefers a gamble f over a gamble g if he both accepts f in exchange for g and disprefers g to f: $f \succ g \Leftrightarrow f \succeq g \wedge g \prec f \Leftrightarrow f - g \in \mathcal{M}_{\succ}$. Two gambles f and g are incomparable when neither of their differences is resolved: $f \asymp g \Leftrightarrow f - g \in \mathcal{M}_{\asymp}$.

Characterization result for gamble relations The nature of these gamble relations follows from the axioms of the accept-reject framework. We here give a translation of these axioms for gamble relations under the form of a characterization: Given gamble relations $\succeq$ and $\prec$ on $\mathcal{L} \times \mathcal{L}$, then these are (5)-equivalent to a model $\mathcal{M}$ in $\mathbb{M}_0$ if and only if for all f, g, and h in $\mathcal{L}$ and $0 < \mu \leq 1$ it holds that
(i) Accept Reflexivity: $f \succeq f$,
(ii) Reject Irreflexivity: $f \nprec f$,
(iii) Accept Transitivity: $f \succeq g \wedge g \succeq h \Rightarrow f \succeq h$,
(iv) Mixed Transitivity: $f \prec g \wedge h \succeq g \Rightarrow f \prec h$,
(v) Mixture independence:
$f \succeq g \Leftrightarrow \mu \cdot f + (1-\mu) \cdot h \succeq \mu \cdot g + (1-\mu) \cdot h$,
$f \prec g \Leftrightarrow \mu \cdot f + (1-\mu) \cdot h \prec \mu \cdot g + (1-\mu) \cdot h$.

So acceptability is a non-strict preorder and dispreference is irreflexive; they are linked together by Mixed Transitivity. Because it is the symmetrization of acceptability $\succeq$, indifference $\simeq$ is an equivalence relation. Preference $\succ$ can be verified to be a strict partial ordering, which makes it suited for decision making. Incomparability $\asymp$ is by definition symmetric.

References

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