Maximin and Maximal Solutions for Linear Programming Problems with Possibilistic Uncertainty

Erik Quaeghebeur, Nathan Huntley,
Keivan Shariatmadar, Gert de Cooman

Ghent University, SYSTeMS Research Group, Belgium

Linear programming problems

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & ax \leq b, \\ & x \geq 0 \end{aligned}$$

Variables x: optimization vector.

Parameters c: objective function coefficient vector,
a: constraint coefficient matrix,
b: constraint coefficient vector.

Linear programming problems *under uncertainty*

$$
\begin{array}{rl}
\text{maximize} & c^T x \\
\text{subject to} & Ax \le B, \\
 & x \ge 0 \\
\text{with} & \text{given uncertainty model for } (A, B)
\end{array}
$$

Variables x: optimization vector.

Parameters c: objective function coefficient vector,

A: constraint coefficient matrix *with uncertain components*,

B: constraint coefficient vector *with uncertain components*.

Independence of components of A and B is assumed.

Linear programming problems under uncertainty

$$\begin{array}{rl} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq B, \\ & x \geq 0 \\ \text{with} & \text{given uncertainty model for } (A, B) \end{array}$$

Variables x: optimization vector.

Parameters c: objective function coefficient vector,
A: constraint coefficient matrix with uncertain components,
B: constraint coefficient vector with uncertain components.

Independence of components of A and B is assumed.

Give meaning by reformulating as a decision problem with *utility functions*

$$G_x := c^T x I_{Ax \leq B} + L I_{Ax \nleq B} = L + (c^T x - L) I_{Ax \leq B}$$

L: penalty value; $L < c^T x$ for 'feasible' x.

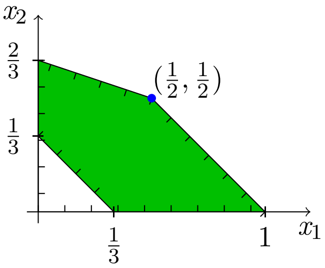
Running example

$$\begin{array}{rl} \text{maximize} & 2x_1 + 3x_2 \\ \text{subject to} & 1x_1 + 3x_2 \leq 2, \\ & 1x_1 + 1x_2 \leq B_2, \\ & -3x_1 - 3x_2 \leq -1, \\ & x_1 \geq 0,\ x_2 \geq 0 \end{array} \quad \equiv \quad \begin{array}{rl} \text{maximize} & c^T x := 2x_1 + 3x_2 \\ \text{subject to} & x \lhd B_2 \end{array}$$

Running example

$$\begin{aligned} \text{maximize} \quad & 2x_1 + 3x_2 \\ \text{subject to} \quad & 1x_1 + 3x_2 \le 2, \\ & 1x_1 + 1x_2 \le B_2, \\ & -3x_1 - 3x_2 \le -1, \\ & x_1 \ge 0,\ x_2 \ge 0 \end{aligned} \quad \equiv \quad \begin{aligned} \text{maximize} \quad & c^T x := 2x_1 + 3x_2 \\ \text{subject to} \quad & x \lhd B_2 \end{aligned}$$

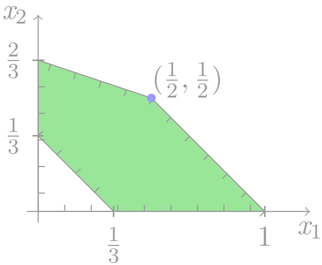
$$\begin{aligned} \text{maximize} \quad & c^T x := 2x_1 + 3x_2 \\ \text{subject to} \quad & x \lhd 1 \end{aligned}$$



Running example

$$
\begin{aligned}
\text{maximize} \quad & 2x_1 + 3x_2 \\
\text{subject to} \quad & 1x_1 + 3x_2 \leq 2, \\
& 1x_1 + 1x_2 \leq B_2, \\
& -3x_1 - 3x_2 \leq -1, \\
& x_1 \geq 0,\ x_2 \geq 0
\end{aligned}
\quad \equiv \quad
\begin{aligned}
\text{maximize} \quad & c^T x := 2x_1 + 3x_2 \\
\text{subject to} \quad & x \lhd B_2
\end{aligned}
$$

$$
\begin{aligned}
\text{maximize} \quad & c^T x := 2x_1 + 3x_2 \\
\text{subject to} \quad & x \lhd 1
\end{aligned}
$$



Penalty value choice $L := 0$ in the running example.

Probabilistic case (probability mass function)

Maximizing expected utility $P(G_x) = L + (c^T x - L)P(Ax \leq B)$

$$\begin{array}{rl} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq B, \\ & x \geq 0 \\ \text{with} & \text{given } p \end{array} \quad \rightarrow \quad \begin{array}{rl} \text{maximize} & P(G_x) \\ \text{subject to} & x \geq 0 \end{array}$$

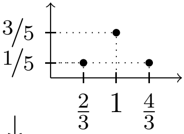
Probabilistic case (probability mass function)

Maximizing expected utility $P(G_x) = L + (c^T x - L) P(Ax \le B)$

$$\begin{array}{rl} \text{maximize} & c^T x \\ \text{subject to} & Ax \le B, \\ & x \ge 0 \\ \text{with} & \text{given } p \end{array} \quad \rightarrow \quad \begin{array}{rl} \text{maximize} & P(G_x) \\ \text{subject to} & x \ge 0 \end{array}$$

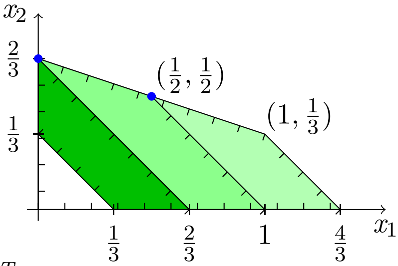
Running example

$$\begin{array}{rl} \text{maximize} & c^T x := 2x_1 + 3x_2 \\ \text{subject to} & x \lhd B_2 \\ \text{with} & p_{B_2} = \end{array}$$



$\downarrow$

$$\begin{array}{rl} \text{maximize} & P(B_2 \ge b) \begin{pmatrix} \text{maximize} & c^T x \\ \text{subject to} & x \lhd b \end{pmatrix} \\ \text{subject to} & b \in \{2/3, 1, 4/3\} \end{array}$$



Optimality criteria for lower & upper previsions

Generalizations of maximizing expected utility for $\underline{P}$ & $\overline{P}$:

Maximinity those $x \geq 0$ are optimal that maximize *lower* expected utility;

$$\underline{P}(G_x) = L + (c^T x - L)\underline{P}(Ax \leq B).$$

Optimality criteria for lower & upper previsions

Generalizations of maximizing expected utility for $\underline{P}$ & $\overline{P}$:

Maximinity: those $x \geq 0$ are optimal that maximize *lower* expected utility;

$$\underline{P}(G_x) = L + (c^T x - L)\underline{P}(Ax \leq B).$$

Maximality: those $x \geq 0$ are optimal that are *undominated* by all other vectors $z \geq 0$ in the sense that

$$\overline{P}(G_x - G_z) = \overline{P}\big((c^T x - L)I_{Ax \leq B} - (c^T z - L)I_{Az \leq B}\big) \geq 0.$$

Optimality criteria for lower & upper previsions

Generalizations of maximizing expected utility for $\underline{P}$ & $\overline{P}$:

Maximinity those $x \geq 0$ are optimal that maximize *lower* expected utility;

$$\underline{P}(G_x) = L + (c^T x - L)\underline{P}(Ax \leq B).$$

Maximality those $x \geq 0$ are optimal that are *undominated* by all other vectors $z \geq 0$ in the sense that

$$\overline{P}(G_x - G_z) = \overline{P}((c^T x - L) I_{Ax \leq B} - (c^T z - L) I_{Az \leq B}) \geq 0.$$

Dominance $x \geq 0$ is *undominated* by $z \geq 0$ in pointwise comparison of utility functions if

$$G_z = G_x \quad \text{or} \quad \max(G_x - G_z) > 0,$$

or, equivalently,

$$c^T x \geq \max_{(Ax \leq B) = (Az \leq B)} c^T z,$$

$$c^T x > \max_{(Ax \leq B) \subset (Az \leq B)} c^T z.$$

Maximin solutions in the interval case

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \le B, \\ & x \ge 0 \\ \text{with} \quad & \underline{a} \le A \le \overline{a}, \\ & \underline{b} \le B \le \overline{b} \end{aligned} \quad \rightarrow \quad \begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & \overline{a} x \le \underline{b}, \\ & x \ge 0 \end{aligned}$$

Maximin solutions in the interval case

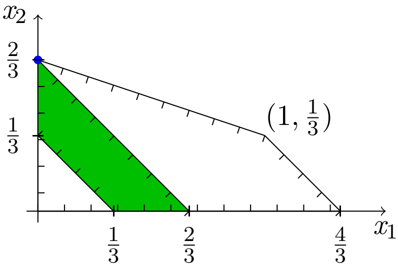
$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq B, \\ & x \geq 0 \\ \text{with} \quad & \underline{a} \leq A \leq \overline{a}, \\ & \underline{b} \leq B \leq \overline{b} \end{aligned} \quad \rightarrow \quad \begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & \overline{a} x \leq \underline{b}, \\ & x \geq 0 \end{aligned}$$

Running example

$$\begin{aligned} \text{maximize} \quad & c^T x := 2x_1 + 3x_2 \\ \text{subject to} \quad & x \lhd B_2 \\ \text{with} \quad & B_2 \in [2/3, 4/3] \end{aligned}$$

$$\downarrow$$

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & x \lhd 2/3 \end{aligned}$$



Maximal solutions in the interval case

$$\begin{aligned}
\text{maximize} \quad & c^T x \\
\text{subject to} \quad & Ax \leq B, \\
& x \geq 0 \\
\text{with} \quad & \underline{a} \leq A \leq \overline{a}, \\
& \underline{b} \leq B \leq \overline{b}
\end{aligned}
\quad \rightarrow \quad
\begin{aligned}
\text{find all} \quad & x \\
\text{subject to} \quad & \underline{a} x \leq \overline{b}, \\
& x \geq 0, \\
& c^T x \geq \max_{\overline{a} z \leq \underline{b}} c^T z, \\
& \text{dominance}
\end{aligned}$$

Maximal solutions in the interval case

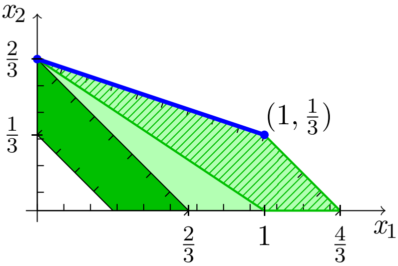
maximize	$c^T x$		find all	x
subject to	$Ax \leq B$,		subject to	$\underline{a}x \leq \overline{b}$,
	$x \geq 0$	$\rightarrow$		$x \geq 0$,
with	$\underline{a} \leq A \leq \overline{a}$,			$c^T x \geq \max_{\overline{a}z \leq \underline{b}} c^T z$,
	$\underline{b} \leq B \leq \overline{b}$			dominance

Running example

maximize	$c^T x := 2x_1 + 3x_2$
subject to	$x \lhd B_2$
with	$B_2 \in [2/3, 4/3]$

$\downarrow$

find all	x
subject to	$x \lhd 4/3$
	$c^T x \geq \max_{z \lhd 2/3} c^T z$,
	$c^T x \geq \max_{1z_1 + 1z_2 \leq 1x_1 + 1x_2} c^T z$ (dominance)



Maximin solutions in the possibilistic case

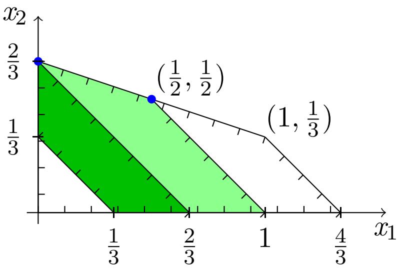
$$\begin{array}{rl} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq B, \\ & x \geq 0 \\ \text{with} & \text{given } \pi \end{array}$$

$$\downarrow$$

$$\begin{array}{rl} \text{maximize} & L + (1-t)\left(\begin{array}{rl} \text{maximize} & c^T x - L \\ \text{subject to} & \overline{a}_t x \leq \underline{b}_t, \\ & x \geq 0 \end{array} \right) \\ \text{subject to} & 0 \leq t < 1 \end{array}$$

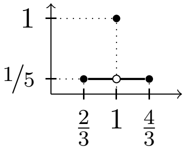
Maximin solutions in the possibilistic case

Running example



$$\begin{array}{rl} \text{maximize} & c^T x := 2x_1 + 3x_2 \\ \text{subject to} & x \lhd B_2 \end{array}$$

with $\pi_{B_2} =$



$$\downarrow$$

$$\begin{array}{rl} \text{maximize} & (1-t)\left(\begin{array}{rl} \text{maximize} & c^T x \\ \text{subject to} & x \lhd \underline{b_2}_t \end{array} \right) \\ \text{subject to} & t \in \{0, 1/5\} \end{array}$$

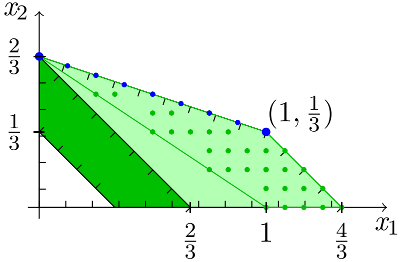
Maximal solutions in the possibilistic case

- No analytical reduction to a standard optimization problem known.
- Numerical approach:
  - Make a grid in the solution set of the corresponding interval case.
  - Compare grid points and remove the dominated ones.
  - This is computationally expensive.

Maximal solutions in the possibilistic case

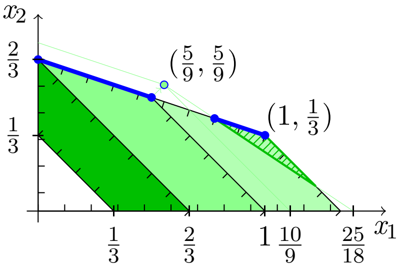
- No analytical reduction to a standard optimization problem known.
- Numerical approach:
  - Make a grid in the solution set of the corresponding interval case.
  - Compare grid points and remove the dominated ones.
  - This is computationally expensive.

Running example (numerical)



Maximal solutions in the possibilistic case

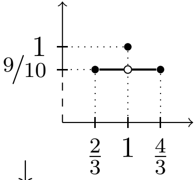
Running example (analytical)



maximize $c^T x := 2x_1 + 3x_2$

subject to $x \lhd B_2$

with $\pi_{B_2} =$



$\downarrow$

find all x

either subject to $x \lhd 1$, $x \not\lhd {}^2\!/_3$

but not $c^T x < \max_{1z_1+1z_2 \leq 1x_1+1x_2} c^T z$

or subject to $x \lhd {}^4\!/_3$, $x \not\lhd 1$,

$c^T x \geq \max_{1z_1+1z_2 \leq 1x_1+1x_2} c^T z$ (dominance)

but not $c^T x < {}^{10}\!/_9 \max_{z \lhd 1} c^T z$ (cf. green-filled dot)

Conclusions

- Problem is very hard in general.
  (Even without dominance.)
- But some specific cases can be tackled, as shown by our results.
- Extension to problems with uncertainty in the goal function . . .
- Using different utility functions . . .