Sets of desirable gambles and their connection to probabilisticly-flavored models for uncertainty

Erik Quaeghebeur

SYSTeMS Research Group, Ghent University

Context & assumptions

Possibility space $\mathcal{X}$ outcomes experiment

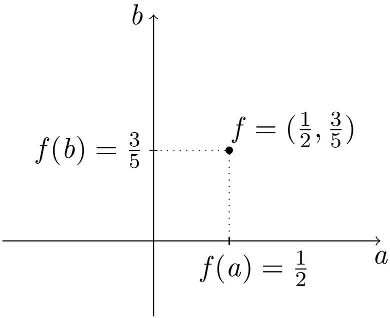
We—an intentional system uncertain about outcome experiment

Goal model our uncertainty/beliefs/information & use this model for reasoning

Gambles payoff depends on outcome,
bounded real-valued function on $\mathcal{X}$,
set of gambles $\mathcal{L}(\mathcal{X})$

Utility linear and precise

Gambles



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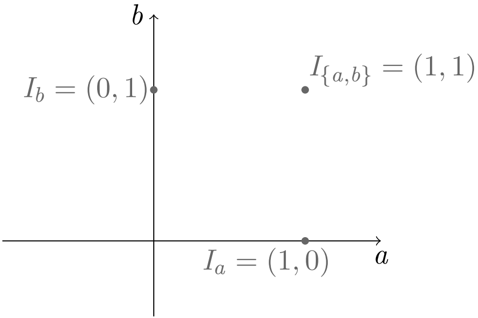

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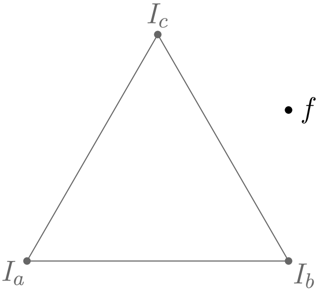

- $f = (-\frac{2}{3}, \frac{5}{6}, \frac{5}{6})$

Desirable gambles

Gamble f desirable when we accept the transaction

(i) the experiment's outcome x is determined

(ii) our capital is changed by $f(x)$

Our uncertainty model set of desirable gambles

Outline

Reasoning about and with sets of desirable gambles

- Rationality criteria
- Assessments avoiding partial (or sure) loss
- Coherent sets of desirable gambles
- Natural extension

Relationships with other, nonequivalent models

Derived coherent sets of desirable gambles

Combining sets of desirable gambles

Partial preference orders

Constructive rationality criteria

It is reasonable to require that a set of desirable gambles $\mathcal{D} \subseteq \mathcal{L}(\mathcal{X})$ satisfies

$$\begin{aligned} \text{Positive scaling:} \quad & \lambda > 0 \Rightarrow \lambda\mathcal{D} = \mathcal{D}, \\ \text{Addition:} \quad & \mathcal{D} + \mathcal{D} = \mathcal{D}. \end{aligned}$$

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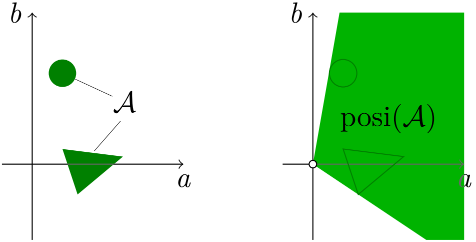
They extend an *assessment* $\mathcal{A} \subseteq \mathcal{L}(\mathcal{X})$ to

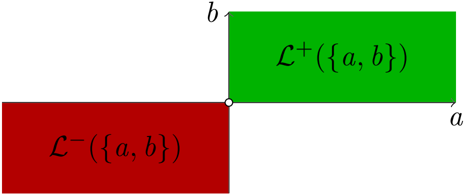
$$\operatorname{posi}(\mathcal{A}) := \left\{ \sum_{k=1}^{n} \lambda_k f_k \colon \lambda_k > 0 \wedge f_k \in \mathcal{L}(\mathcal{X}) \wedge n \in \mathbb{N} \right\}$$



Constraining rationality criteria

Comparing gambles the ordinary vector ordering is defined by

$$f \geq g \Leftrightarrow (f-g) \in \mathcal{L}_0^+(\mathcal{X}) \Leftrightarrow \inf(f-g) \geq 0$$
$$f > g \Leftrightarrow (f-g) \in \mathcal{L}^+(\mathcal{X}) \Leftrightarrow \inf(f-g) \geq 0 \wedge \sup(f-g) > 0$$



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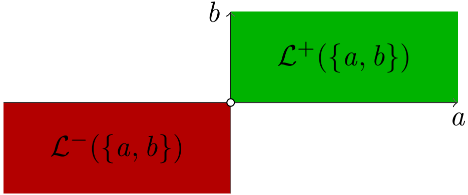

It is reasonable to require that a set of desirable gambles $\mathcal{D} \subseteq \mathcal{L}(\mathcal{X})$ satisfies

Accepting partial gain:	$\mathcal{L}^+(\mathcal{X}) \subseteq \mathcal{D}$
Avoiding partial loss:	$\mathcal{D} \cap \mathcal{L}^-(\mathcal{X}) = \emptyset$

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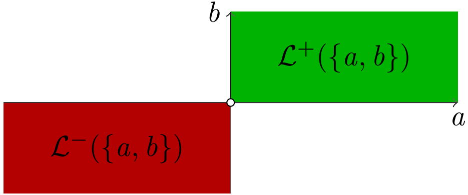

If $\mathcal{D} \subseteq \mathcal{L}(\mathcal{X})$ accepts partial gain and avoids partial loss, then it also satisfies

Accepting sure gain: $\operatorname{int}(\mathcal{L}^+(\mathcal{X})) \subseteq \mathcal{D}$

Avoiding sure loss: $\mathcal{D} \cap \operatorname{int}(\mathcal{L}^-(\mathcal{X})) = \emptyset$

Assessments & partial loss

An assessment $\mathcal{A} \subseteq \mathcal{L}(\mathcal{X})$ *avoids partial loss* iff

$$\mathrm{posi}(\mathcal{A}) \cap \mathcal{L}^{-}(\mathcal{X}) = \emptyset$$

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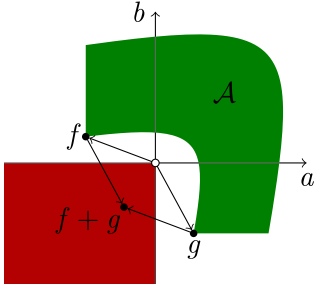
An assessment $\mathcal{A} \subseteq \mathcal{L}(\mathcal{X})$ *incurs partial loss* iff

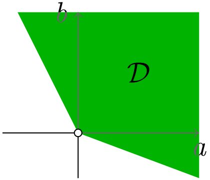
$$\operatorname{posi}(\mathcal{A}) \cap \mathcal{L}^{-}(\mathcal{X}) \neq \emptyset$$



Coherent sets of desirable gambles

Coherence A set of desirable gambles $\mathcal{D} \subseteq \mathcal{L}(\mathcal{X})$ is coherent if it satisfies all four rationality criteria.

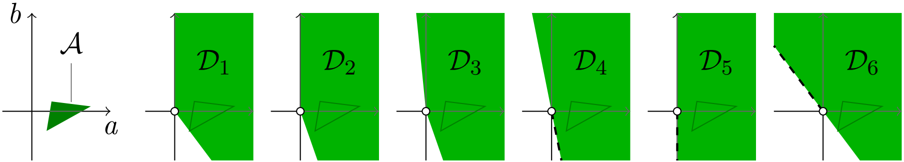
Geometry It is a convex cone containing the positive orthant $\mathcal{L}^+(\mathcal{X})$, but excluding the negative orthant $\mathcal{L}^-(\mathcal{X})$.



Set of coherent sets $\mathbb{D}(\mathcal{X})$

Coherent extensions

Coherent extensions of an assessment $\mathcal{A} \subseteq \mathcal{L}(\mathcal{X})$ Any encompassing coherent set of desirable gambles



Set of coherent extensions $\mathbb{D}_{\mathcal{A}} \coloneqq \{\mathcal{D} \in \mathbb{D}(\mathcal{X}) \colon \mathcal{A} \subseteq \mathcal{D}\}$

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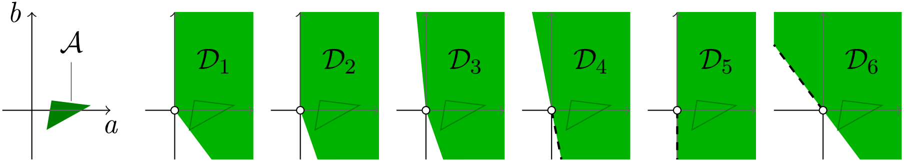

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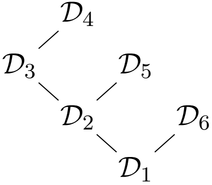
Inclusion based partial order of extensions that are more/less *committal*



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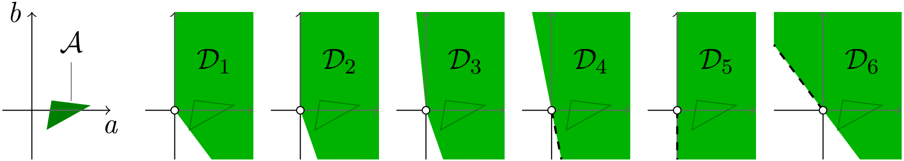

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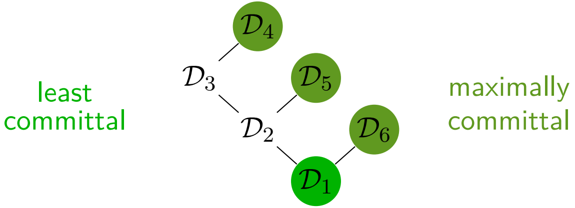
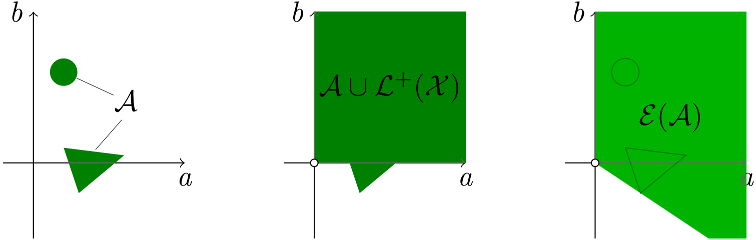


Natural extension

Given the constructive rationality criteria and accepting partial gains, there is a *natural extension* of an assessment $\mathcal{A} \subseteq \mathcal{L}(\mathcal{X})$:

$$\mathcal{E}(\mathcal{A}) := \operatorname{posi}(\mathcal{A} \cup \mathcal{L}^+(\mathcal{X}))$$



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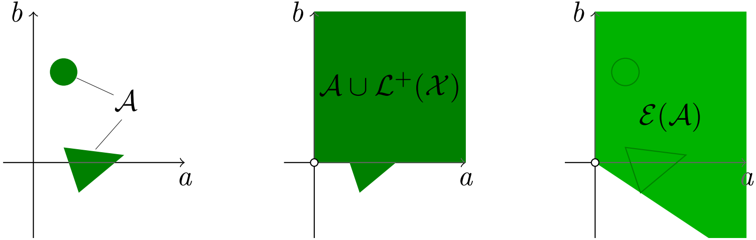

Natural Extension Theorem

The natural extension $\mathcal{E}(\mathcal{A})$ of $\mathcal{A} \subseteq \mathcal{L}(\mathcal{X})$ coincides with its least committal coherent extension $\bigcap \mathbb{D}_{\mathcal{A}}$ if and only if $\mathcal{A}$ avoids partial loss.

Natural extension is the prime tool for *deductive inference* in desirability.

Outline

Reasoning about and with sets of desirable gambles

Relationships with other, nonequivalent models
- Linear previsions
- Credal sets
- To lower & upper previsions
- From lower previsions

Derived coherent sets of desirable gambles

Combining sets of desirable gambles

Partial preference orders

Linear previsions

Linear previsions ...

- are positive linear normed expectation operators
- provide fair prices for gambles in $\mathcal{L}(\mathcal{X})$
- are equivalent to (finitely additive) probability measures and, on finite $\mathcal{X}$, to probability mass functions

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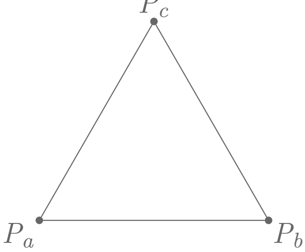
- belong to the closed convex set $\mathbb{P}(\mathcal{X})$ which is, for finite $\mathcal{X}$, the unit simplex spanned by the degenerate previsions (or $\{0, 1\}$-valued probability mass functions)



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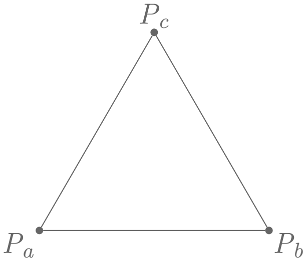

- provide probabilities for events, as fair prices for their indicators

From linear previsions to sets of desirable gambles

Given a linear prevision $P \in \mathbb{P}(\mathcal{X})$, gambles with a strictly positive fair price are strictly desirable:

$$\mathcal{D}_P \coloneqq \mathcal{E}(\mathcal{A}_P), \quad \text{with} \quad \mathcal{A}_P \coloneqq \{f \in \mathcal{L}(\mathcal{X})\colon P(f) > 0\}$$

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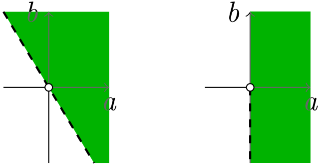
Observations:

- $\{f \in \mathcal{L}(\mathcal{X})\colon P(f) = 0\}$ is a linear subspace of $\mathcal{L}(\mathcal{X})$
- So $\mathcal{A}_P$ is an open halfspace
- Except in a few borderline cases, so is $\mathcal{D}_P$



From credal sets to sets of desirable gambles

A credal set is a set of linear previsions

Given a credal set $\mathcal{M} \subseteq \mathbb{P}(\mathcal{X})$, gambles with a strictly positive fair price for every linear prevision in the credal set are strictly desirable:

$$\mathcal{D}_{\mathcal{M}} \coloneqq \mathcal{E}(\mathcal{A}_{\mathcal{M}}), \quad \text{with} \quad \mathcal{A}_{\mathcal{M}} \coloneqq \{f \in \mathcal{L}(\mathcal{X}) \colon (\forall P \in \mathcal{M} : P(f) > 0)\} = \bigcap_{P \in \mathcal{M}} \mathcal{A}_P$$

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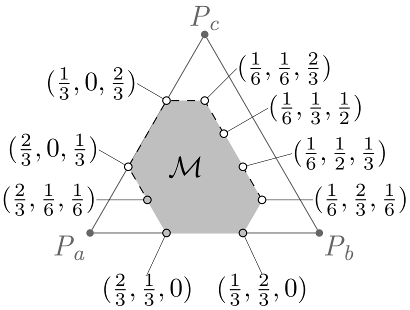
Observations:

- Each prevision gives rise to a linear constraint in gamble space
- Constraints from linear previsions strictly in the convex hull of $\mathcal{M}$ are redundant
- So the border structure of $\mathcal{M}$ is uniquely important

From credal sets to sets of desirable gambles: example



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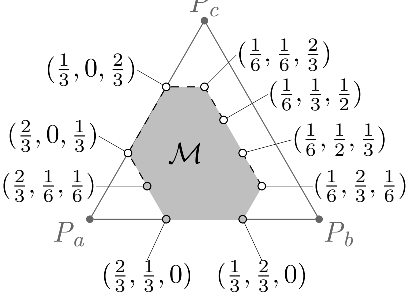

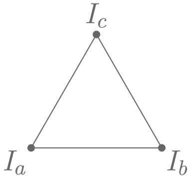

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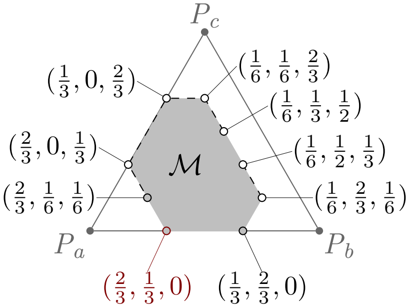
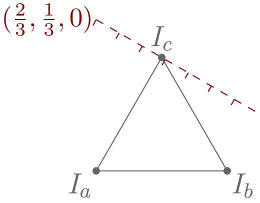


$$\frac{2}{3}f(a) + \frac{1}{3}f(b) > 0$$



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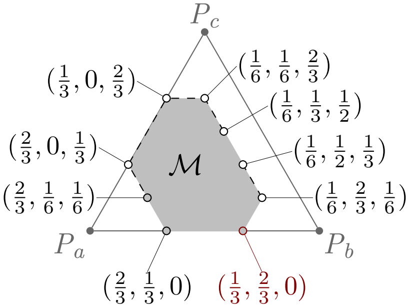

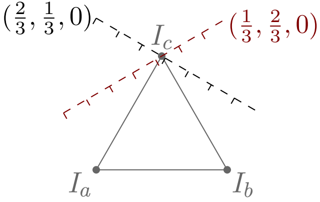

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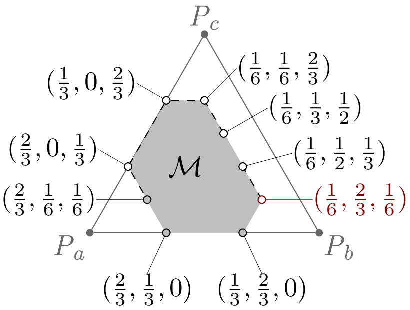

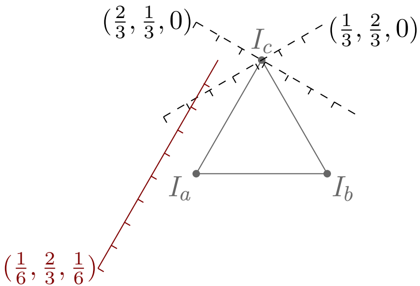

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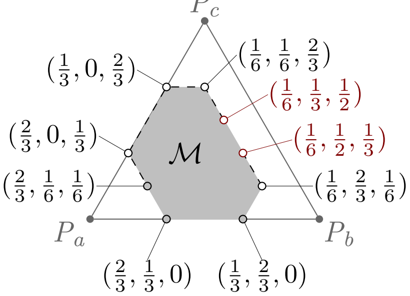

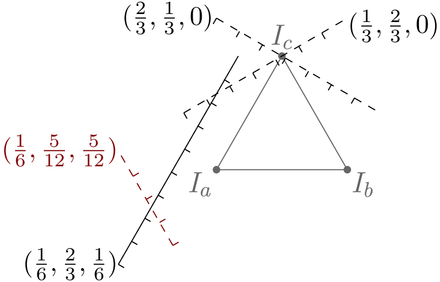

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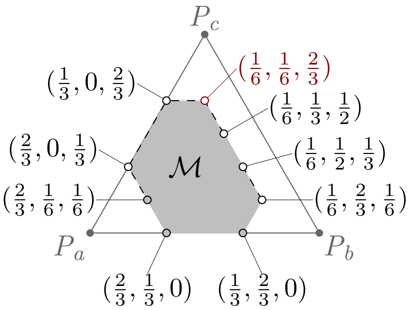

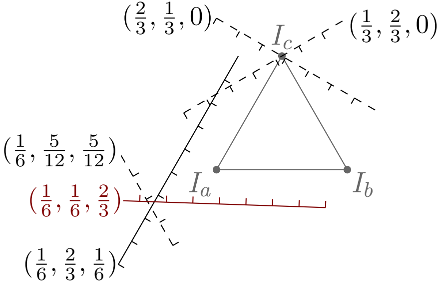

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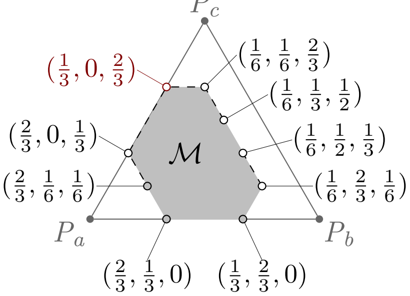

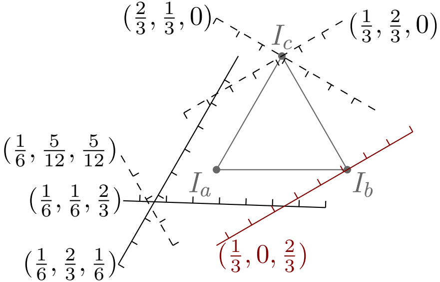

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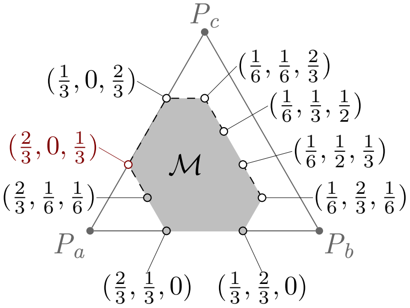

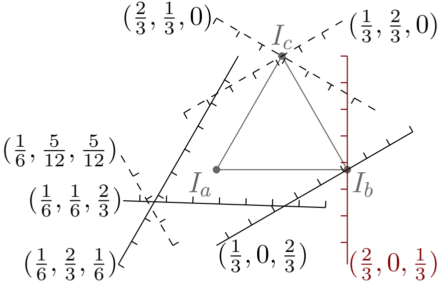

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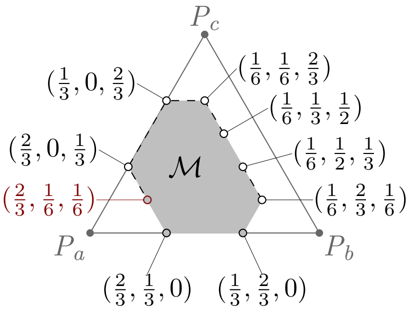

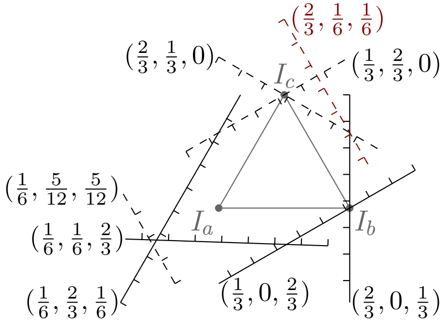

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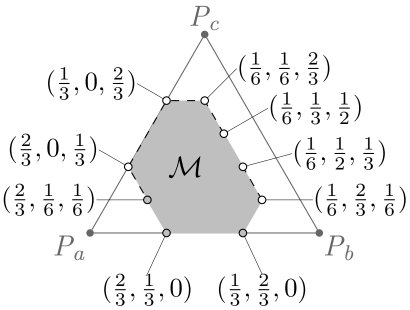

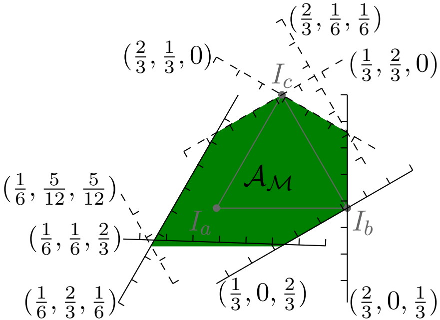

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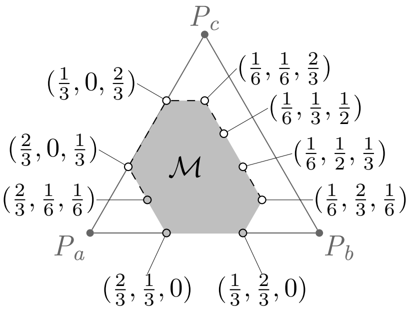

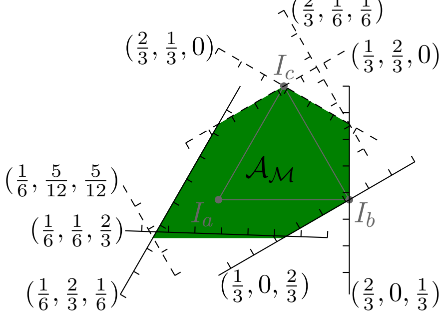

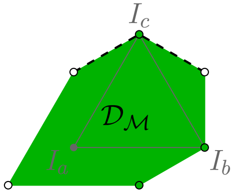

From desirable gambles to credal sets

Given a coherent set of strictly desirable gambles $\mathcal{D} \subset \mathcal{L}(\mathcal{X})$,
we derive the associated credal set as follows:

$$\mathcal{M}_{\mathcal{D}} \coloneqq \bigcap_{f \in \mathcal{D}} \{P \in \mathbb{P}(\mathcal{X})\colon P(f) \geq 0\}.$$

Credal Set Proposition

The credal set $\mathcal{M}_{\mathcal{D}} \subseteq \mathbb{P}(\mathcal{X})$ associated to a coherent set of desirable gambles $\mathcal{D} \subset \mathcal{L}(\mathcal{X})$ is closed and convex.

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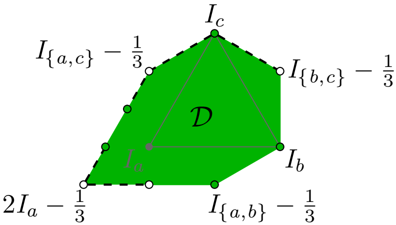

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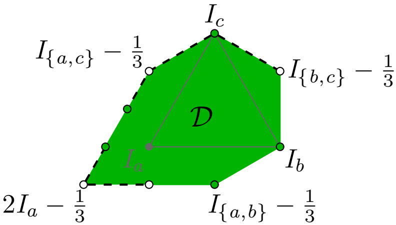

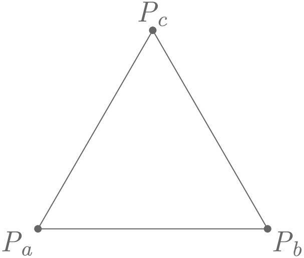

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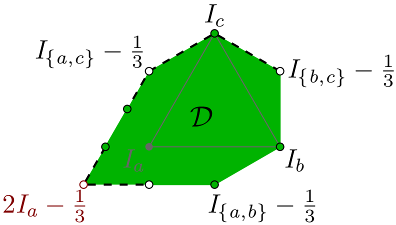

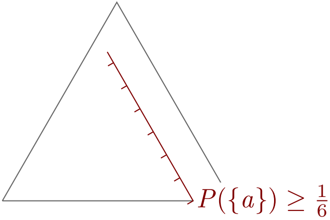

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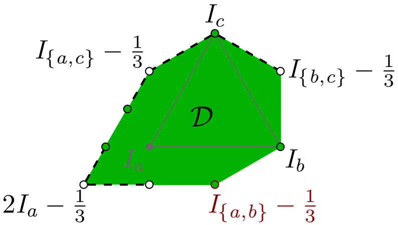

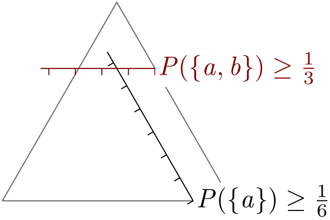

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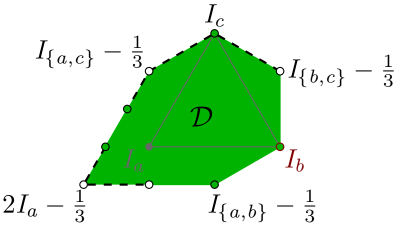

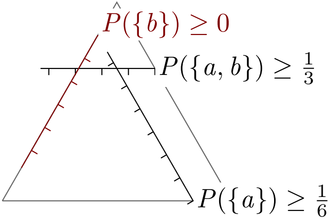

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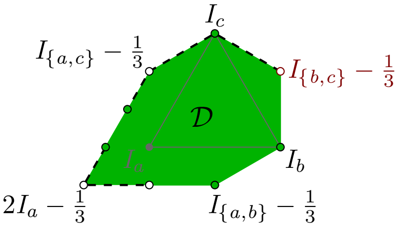

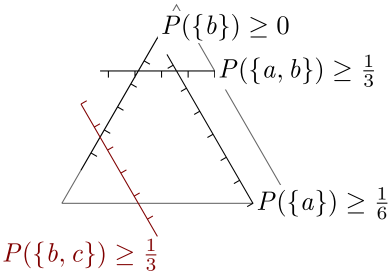

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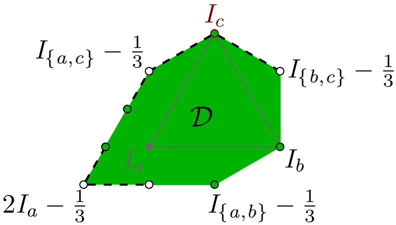

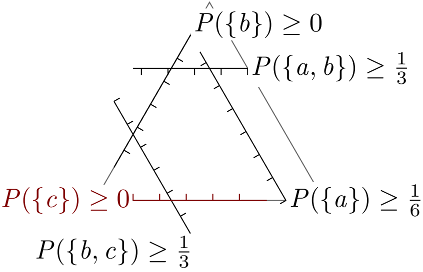

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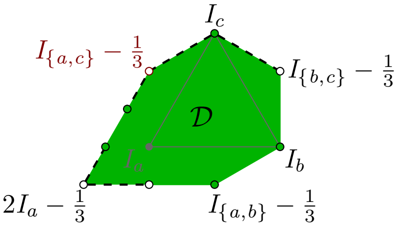

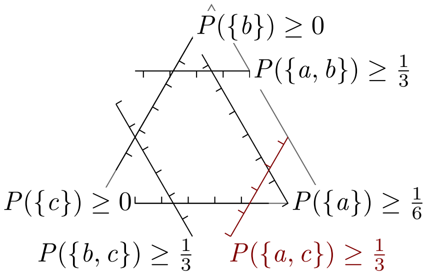

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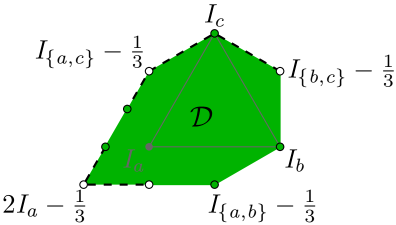

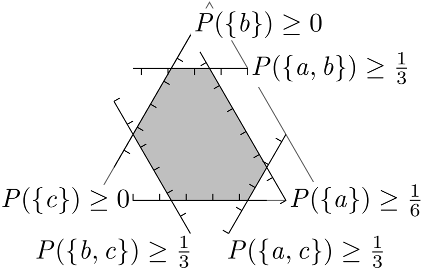

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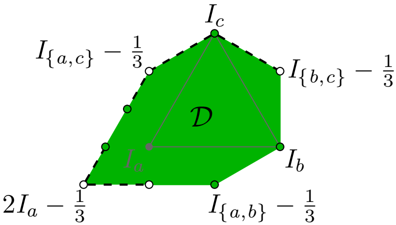

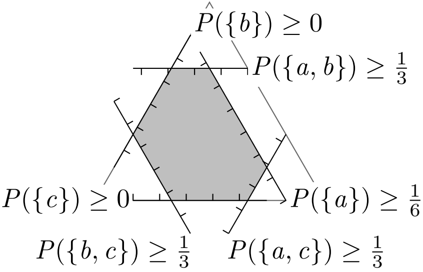

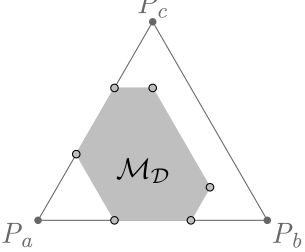

Lower & upper previsions

Lower previsions ...

- are positive superlinear normed expectation operators
- provide supremum acceptable buying prices for gambles in $\mathcal{L}(\mathcal{X})$
- provide lower probabilities for events

Upper previsions ...

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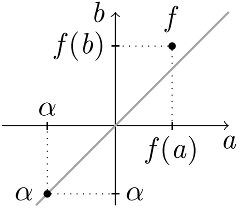
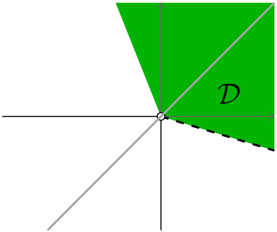
Prices can be seen as constant gambles, which are trivially linearly ordered



From sets of desirable gambles to lower & upper previsions

Given a coherent set of strictly desirable gambles $\mathcal{D} \subset \mathcal{L}(\mathcal{X})$, we use comparisons with constant gambles to derive lower and upper previsions:



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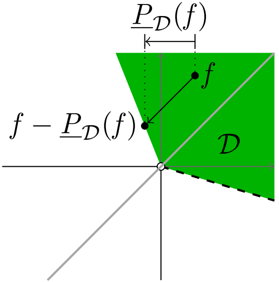

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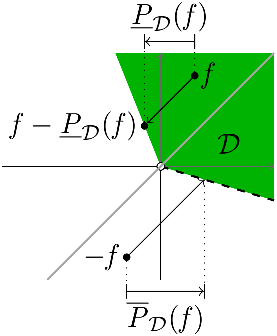

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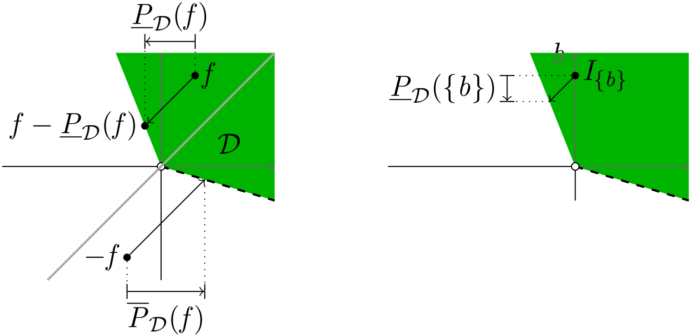

Conjugacy: $\overline{P}_{\mathcal{D}}(f) = -\underline{P}_{\mathcal{D}}(-f)$ and $\overline{P}_{\mathcal{D}}(A) = 1 - \underline{P}_{\mathcal{D}}(A^c)$

From lower previsions to sets of desirable gambles

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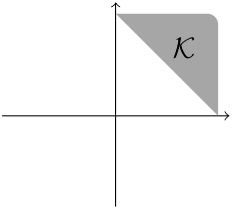
Use marginal as marginally desirable gambles, i.e., border gambles:

$$\mathcal{D}_{\underline{P}} \coloneqq \mathcal{E}(\mathcal{A}_{\underline{P}}) \quad \text{with} \quad \mathcal{A}_{\underline{P}} \coloneqq \mathcal{G}_{\underline{P}} + \mathbb{R}^+ \quad \text{and} \quad \mathcal{G}_{\underline{P}} \coloneqq G_{\underline{P}}(\mathcal{K})$$



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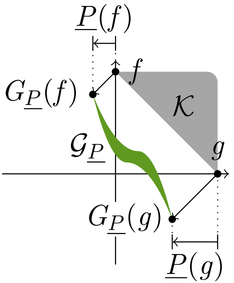

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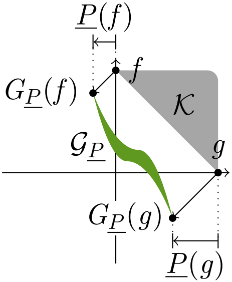

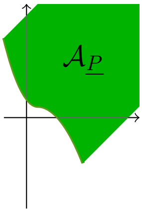

Translating desirability concepts to lower previsions

Avoiding sure loss for a lower prevision $\underline{P}$ on $\mathcal{K} \subseteq \mathcal{L}(\mathcal{X})$ corresponds to $\mathcal{A}_{\underline{P}}$ avoiding sure (or partial) loss:

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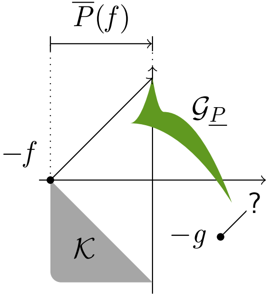

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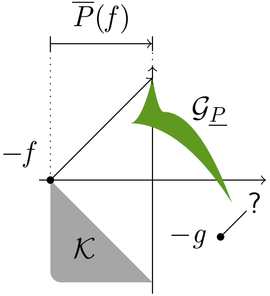

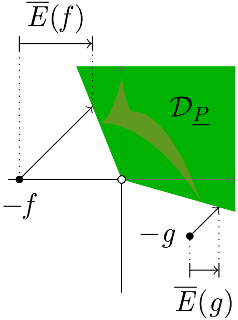

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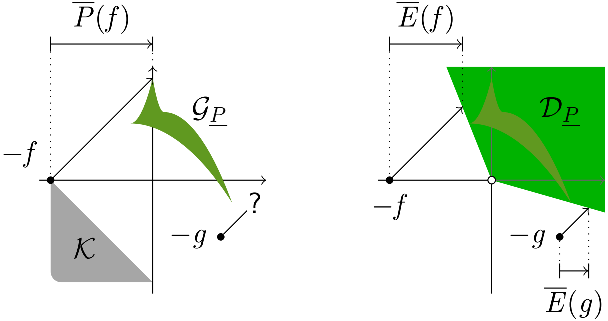

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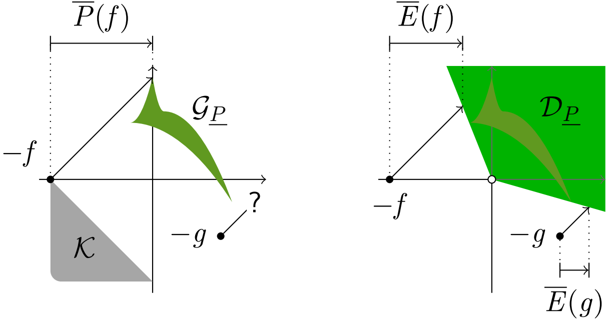

Coherence for lower previsions $\underline{P}$ on $\mathcal{K} \subseteq \mathcal{L}(\mathcal{X})$ corresponds to coherence of $\mathcal{D}_{\underline{P}}$:

$$\forall f \in \mathcal{G}_{\underline{P}} : \forall g \in \mathrm{posi}(\mathcal{G}_{\underline{P}}) : \sup(g - f) \geq 0$$

Outline

Reasoning about and with sets of desirable gambles

Relationships with other, nonequivalent models

Derived coherent sets of desirable gambles

- Gamble space transformations
- Conditional sets of desirable gambles
- Conditional lower previsions
- Marginal sets of desirable gambles

Combining sets of desirable gambles

Partial preference orders

Transformation of a set of desirable gambles

$$\mathcal{D}_\Gamma := \{h \in \mathcal{L}(\mathcal{Z})\colon \Gamma h \in \mathcal{D}\}$$

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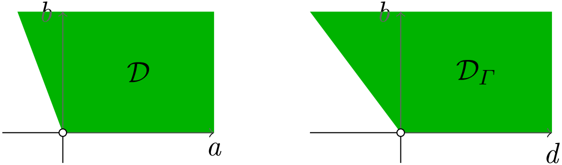
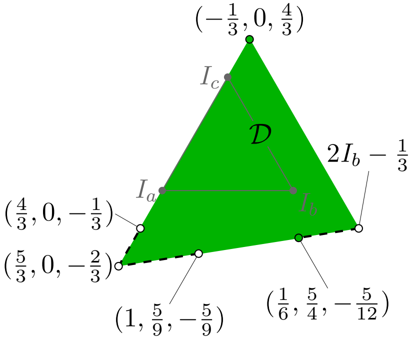


- $\Gamma : \mathcal{L}(\{d, b\}) \to \mathcal{L}(\{a, b\})$
- $(\Gamma h)(a) = \frac{1}{2}h(d)$ and $(\Gamma h)(b) = h(b)$

Taking a slice of a set of desirable gambles



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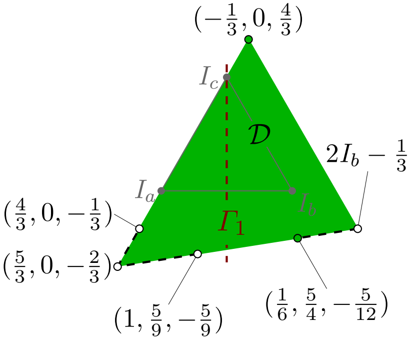

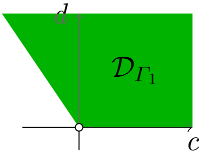

- $\Gamma_1 : \mathcal{L}(\{c, d\}) \to \mathcal{L}(\{a, b, c\})$
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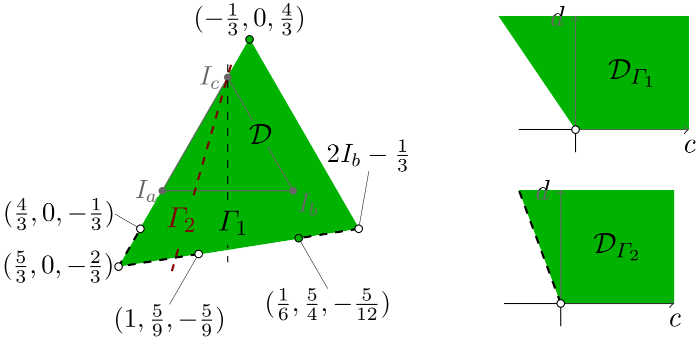

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- $(\Gamma_2 h)(a) = \frac{3}{4}h(d)$, $(\Gamma_2 h)(b) = \frac{1}{4}h(d)$ and $(\Gamma_2 h)(c) = h(c)$

Conditional sets of desirable gambles

Conditioning event $B \subseteq \mathcal{X}$ is what the experiment's outcome is assumed to belong to

Contingent gambles are those for which, if B does not occur, status quo is maintained

Transformation $\nmid_{B^c}$ maps gambles on B to contingent gambles on $\mathcal{X}$:

$$(\nmid_{B^c} h)(x) = \begin{cases} h(x), & x \in B, \\ 0, & x \in B^c, \end{cases}$$

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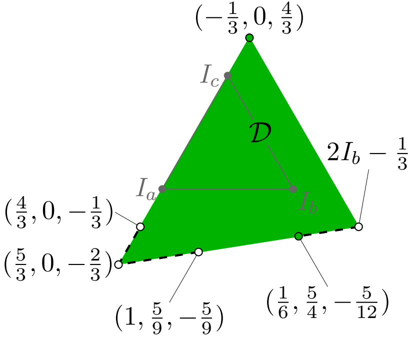
$$\mathcal{D}|B \coloneqq \mathcal{D}_{\nmid_{B^c}} = \{h \in \mathcal{L}(B) \colon \nmid_{B^c} h \in \mathcal{D}\}$$

Conditional sets of desirable gambles: example



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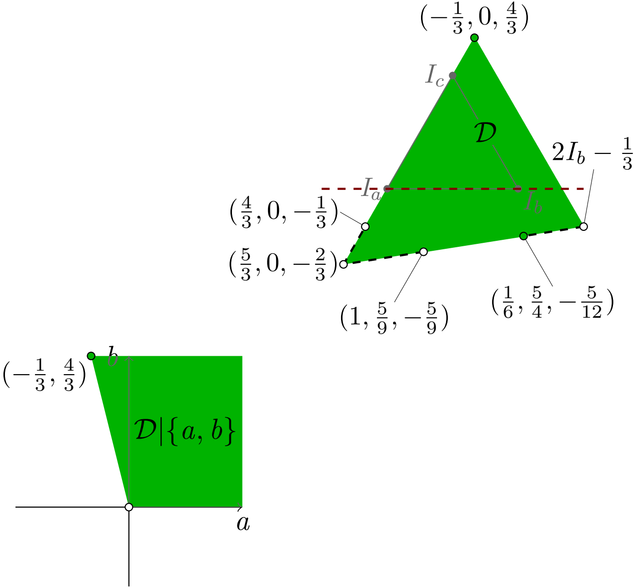

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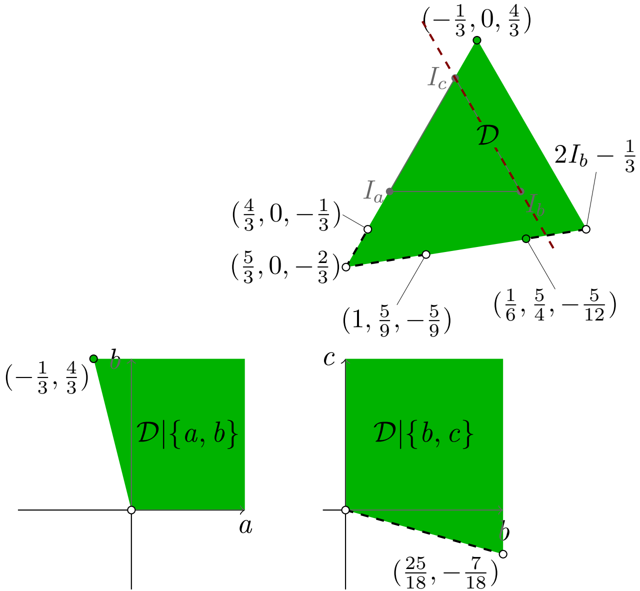

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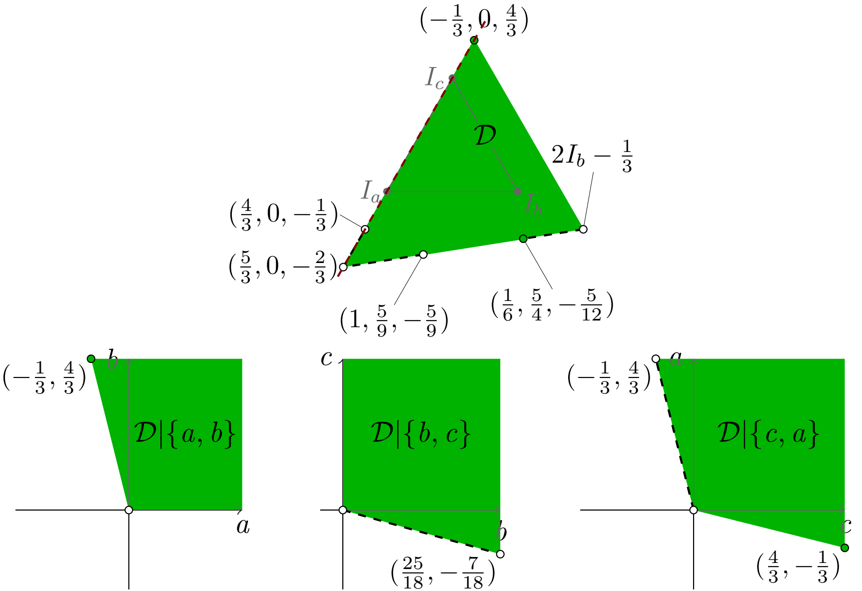

Natural versus regular extension

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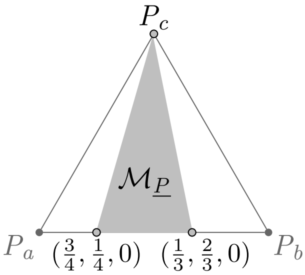

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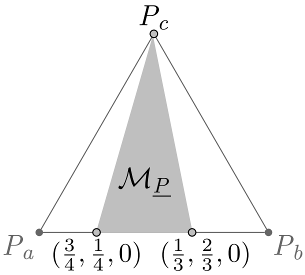

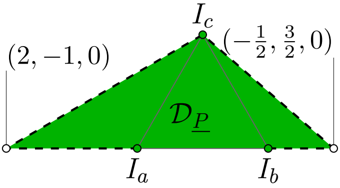

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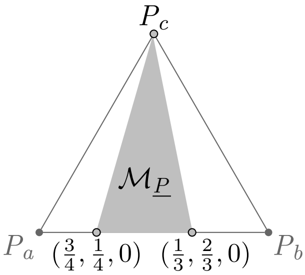

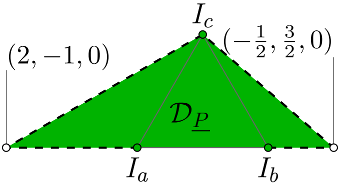

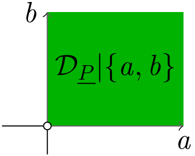

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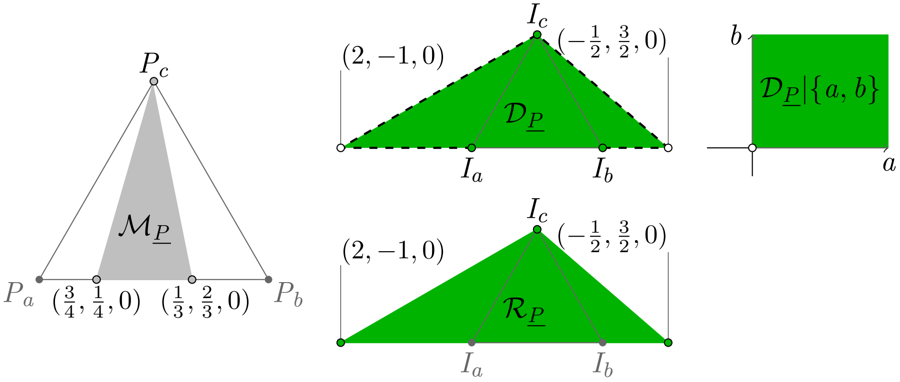

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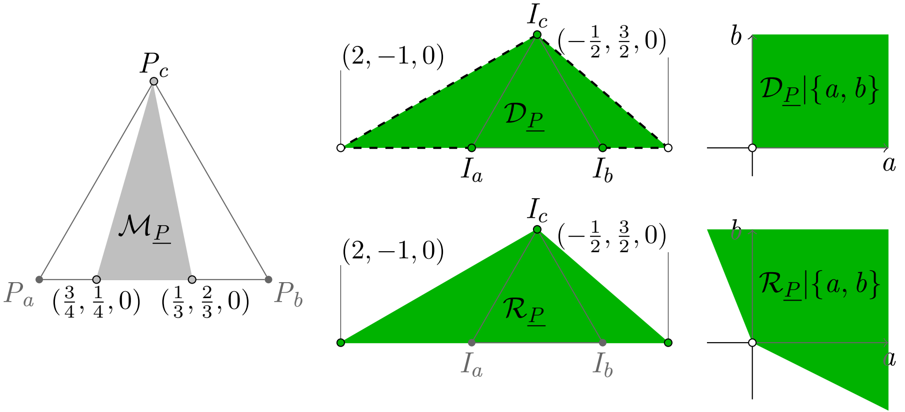

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Marginal sets of desirable gambles

Cartesian product possibility space $\mathcal{X} = \mathcal{Y} \times \mathcal{Z}$, focus on $\mathcal{Y}$-component (ignore $\mathcal{Z}$-component)

Cylindrical extension $\uparrow_{\mathcal{Z}}$ maps gambles from the source gamble space to its cartesian product with $\mathcal{L}(\mathcal{Z})$:

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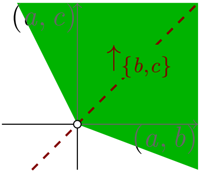
Marginal set of desirable gambles Given a set of desirable gambles $\mathcal{D} \subseteq \mathcal{L}(\mathcal{Y} \times \mathcal{Z})$, its $\mathcal{Y}$-*marginal* is

$$\mathcal{D}{\downarrow}\mathcal{Y} := \mathcal{D}_{\uparrow_{\mathcal{Z}}} = \{h \in \mathcal{L}(\mathcal{Y}) \colon \uparrow_{\mathcal{Z}} h \in \mathcal{D}\}$$



Outline

Reasoning about and with sets of desirable gambles

Relationships with other, nonequivalent models

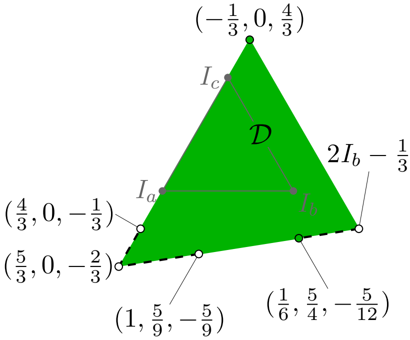
Derived coherent sets of desirable gambles

Combining sets of desirable gambles

- Joining compatible individuals
- Marginal extension

Partial preference orders

Combining sets of desirable gambles: example



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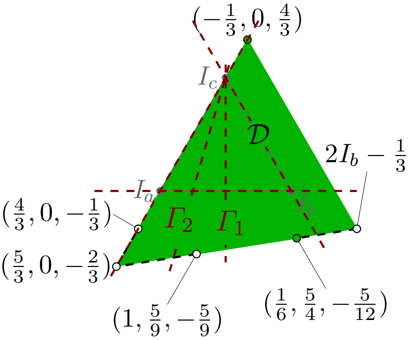

$$
\begin{aligned}
\mathcal{A} \coloneqq \Gamma_1(\mathcal{D}_{\Gamma_1}) \cup \Gamma_2(\mathcal{D}_{\Gamma_2}) &\cup \upharpoonleft_{\{c\}}(\mathcal{D}|\{a, b\}) \\
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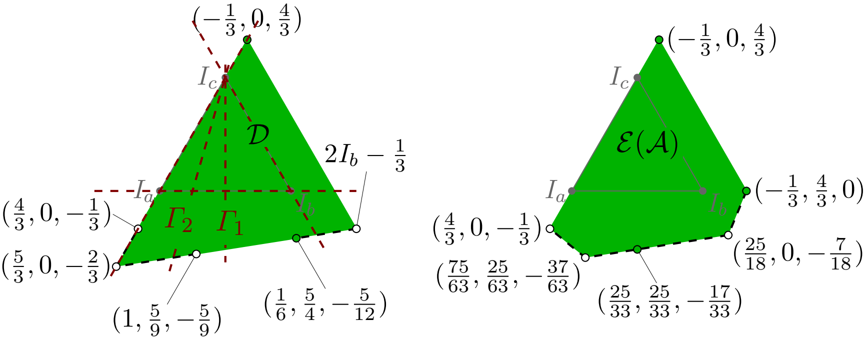

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Given a partition $\mathcal{B}$ of $\mathcal{X}$, a coherent $\mathcal{B}$-marginal $\mathcal{D}_{\mathcal{B}} \subset \mathcal{L}(\mathcal{B})$, and separately coherent conditional sets of desirable gambles $\mathcal{D}|B \subset \mathcal{L}(B)$, $B \in \mathcal{B}$, then their combination $\mathcal{D} := \mathcal{E}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{X})$, with $\mathcal{A} := \mathit{\Gamma}_{\mathcal{B}}(\mathcal{D}_{\mathcal{B}}) \cup \bigcup_{B \in \mathcal{B}} \upharpoonleft_{B^c}(\mathcal{D}|B)$, is coherent as well.

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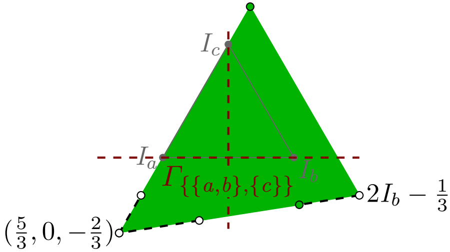

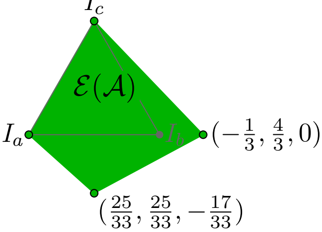

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Partial preference orders

- Strict preference
- Nonstrict preference
- Examples

Partial strict preference order

Strict preference $f \succ g$ if we are eager to exchange g for f

Partial . . . order The order does not have to be complete,
$f \not\succ g \wedge g \not\succ f$ is possible

Strict desirability is strict preference over status quo, the zero gamble 0:

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Rationality criteria for strict preference relations $\succ$ on $\mathcal{L}(\mathcal{X}) \times \mathcal{L}(\mathcal{X})$:

Irreflexivity:	$f \not\succ f$
Transitivity:	$f \succ g \wedge g \succ h \Rightarrow f \succ h$
Mix-indep.:	$0 < \mu \leq 1 \Rightarrow \big(f \succ g \Leftrightarrow \mu f + (1-\mu)h \succ \mu g + (1-\mu)h\big)$
Monotonicity:	$f > g \Rightarrow f \succ g$

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Strengthening coherence criteria for sets of desirable gambles $\mathcal{D}$:

Avoiding nonpositivity: $0 \notin \mathcal{D}$

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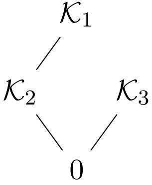
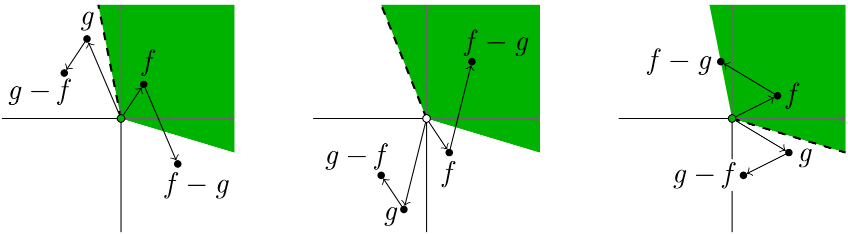


Example:

- $\equiv$-equivalence classes $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$
- intransitivity of $\bowtie$:
  $\mathcal{K}_1 \bowtie \mathcal{K}_3$ and $\mathcal{K}_3 \bowtie \mathcal{K}_2$, but $\mathcal{K}_1 \trianglerighteq \mathcal{K}_2$

Strict and nonstrict preferences: examples



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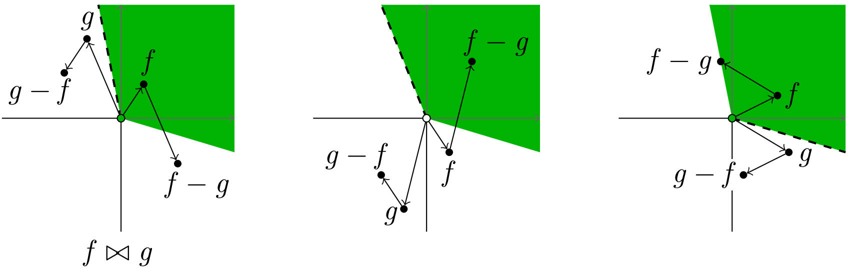

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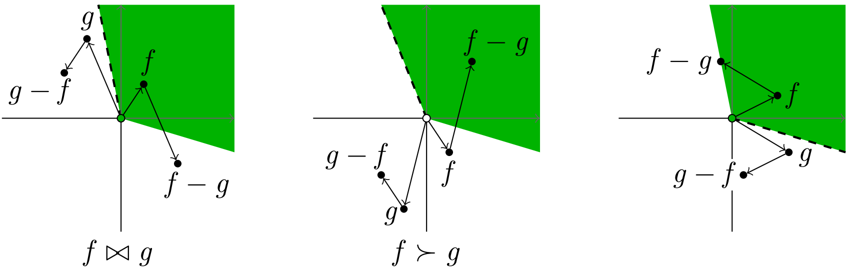

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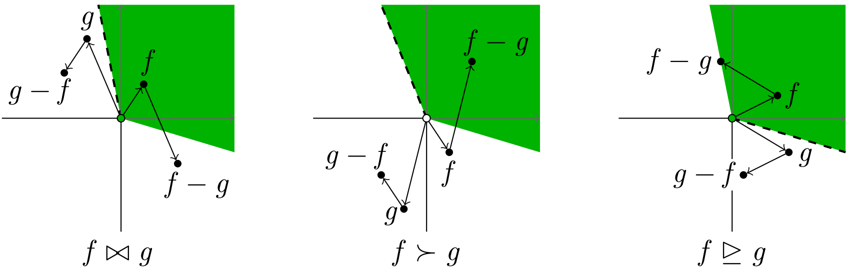

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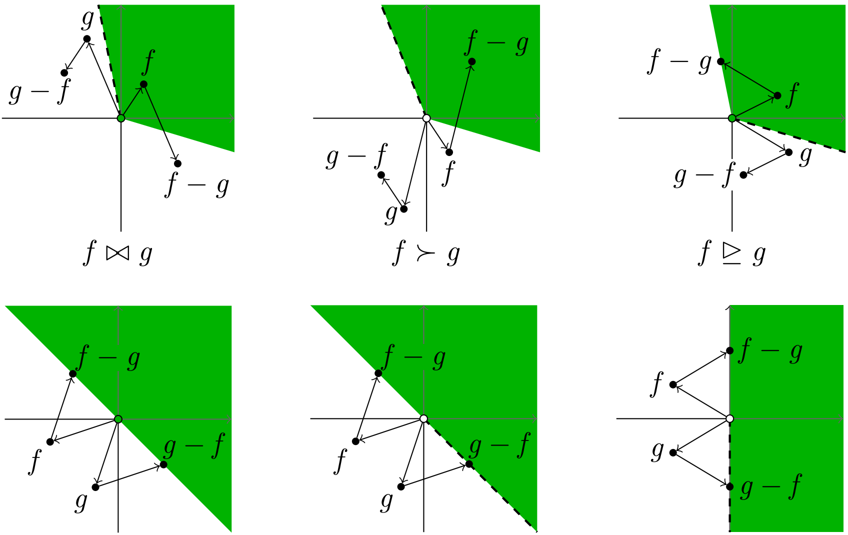

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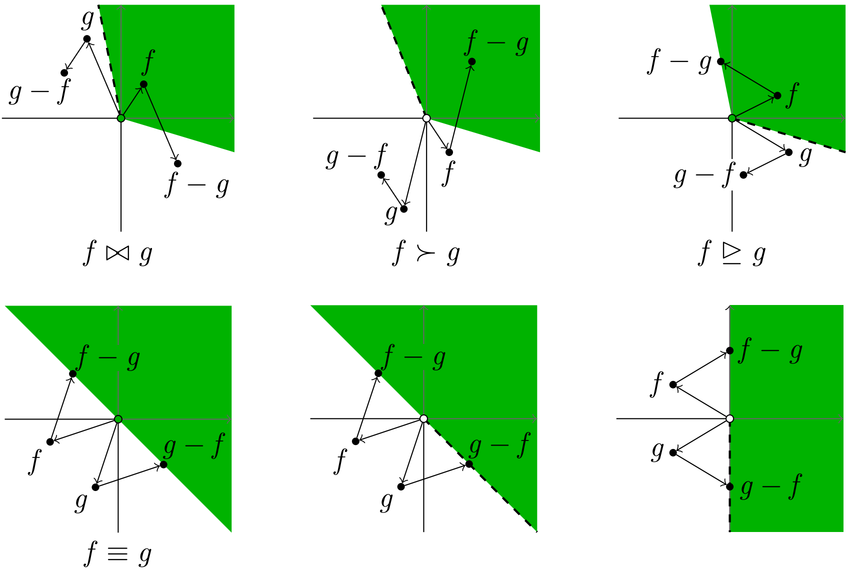

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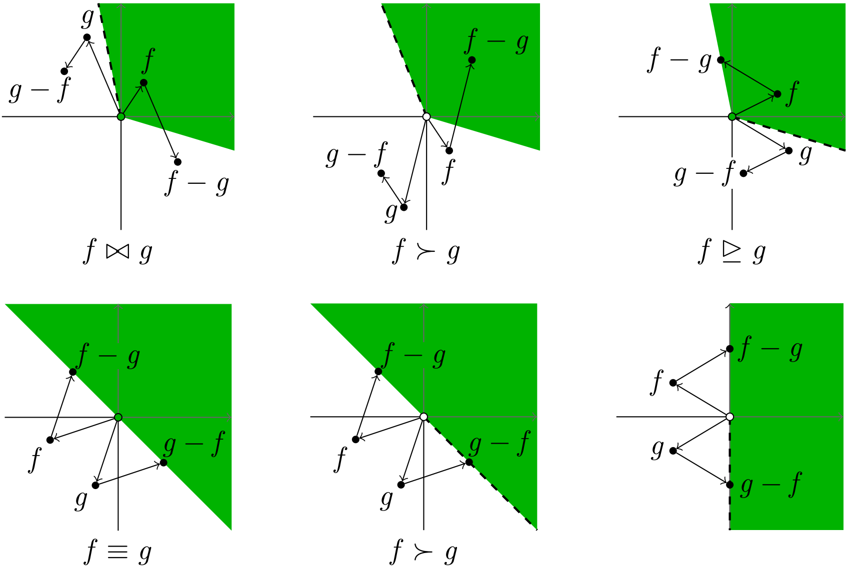

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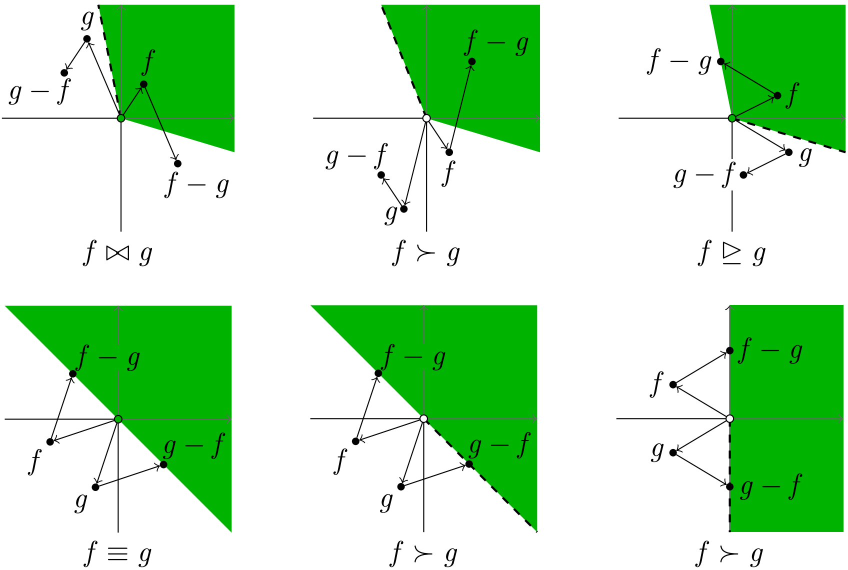

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Full section outline

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