Finitary characterizations of sets of lower previsions

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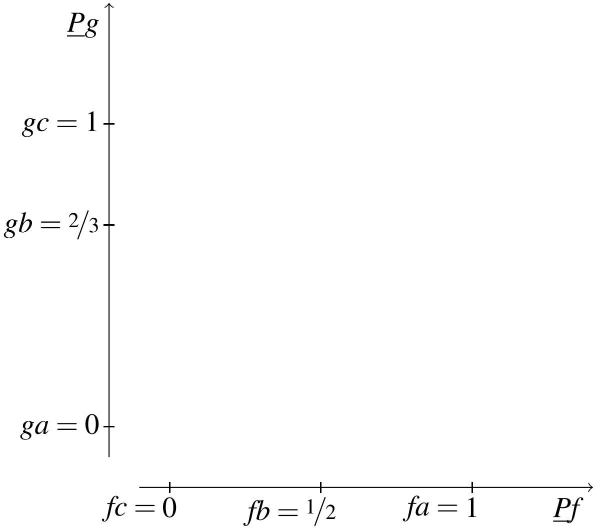
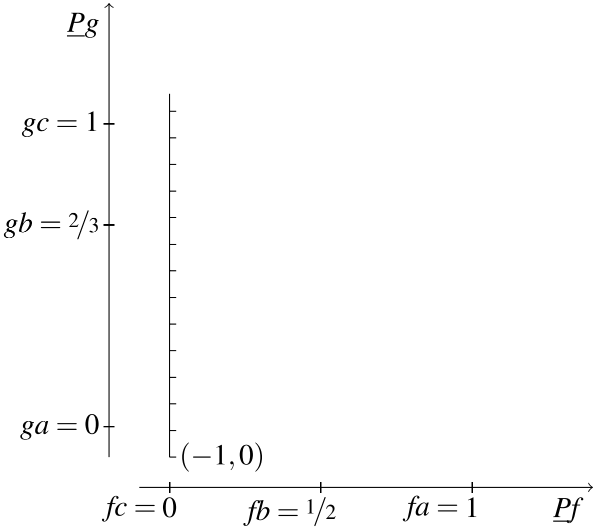
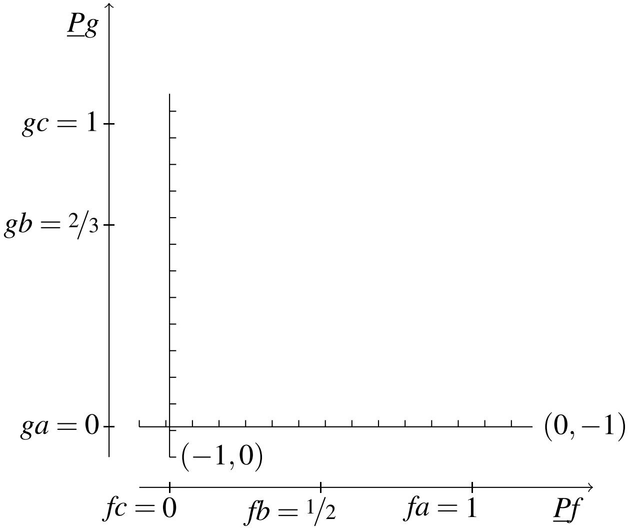
$\underline{P}$ on $\mathscr{K}$ is coherent iff $\sum_{h \in \mathscr{K}} \lambda_h \cdot \underline{P}h \leq \max \sum_{h \in \mathscr{K}} \lambda_h \cdot h$
for all λ in $\mathbb{R}^{\mathscr{K}}$ with at most one strictly negative component



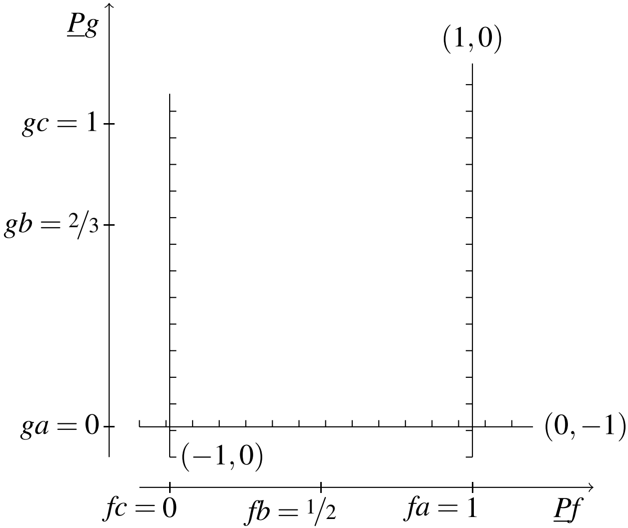
$$\lambda_f \cdot \underline{P}f + \lambda_g \cdot \underline{P}g \leq \max\{\lambda_f, \lambda_f \cdot {}^1\!/\!_2 + \lambda_g \cdot {}^2\!/\!_3, \lambda_g\}$$

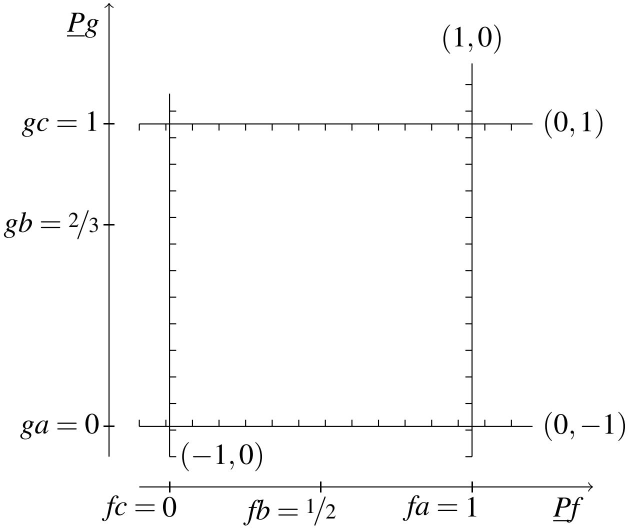


$$-\underline{P}f \leq \max\{-1, -1/2, 0\} = 0$$

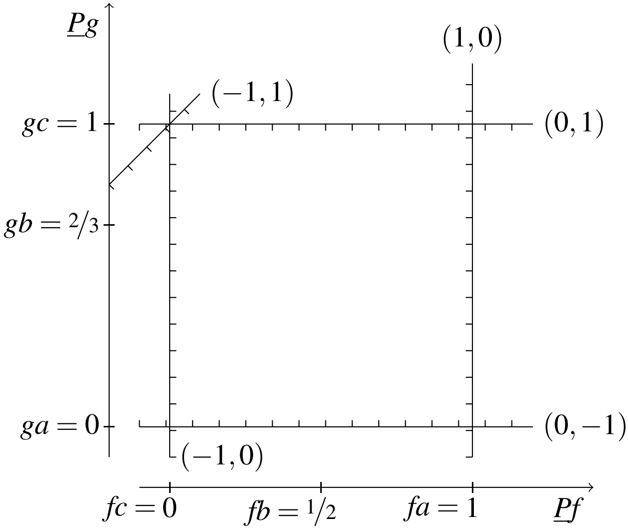


$\underline{P}g$
$gc = 1$
$gb = 2/3$
$ga = 0$
$(0, -1)$
$(-1, 0)$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$

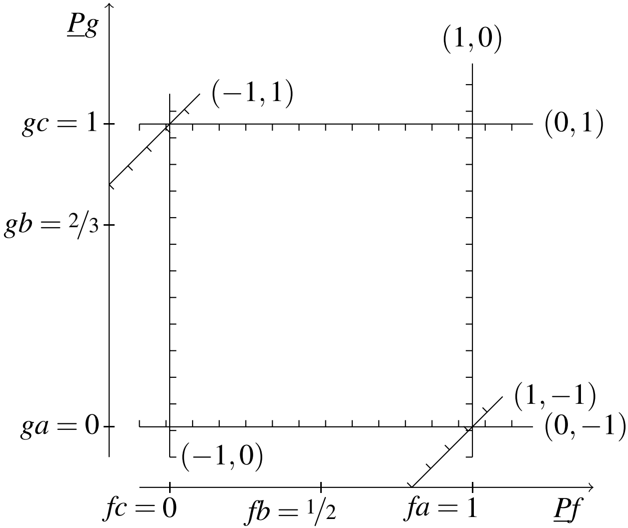

$\underline{P}g$
$(1,0)$
$gc = 1$
$gb = 2/3$
$ga = 0$
$(0,-1)$
$(-1,0)$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$



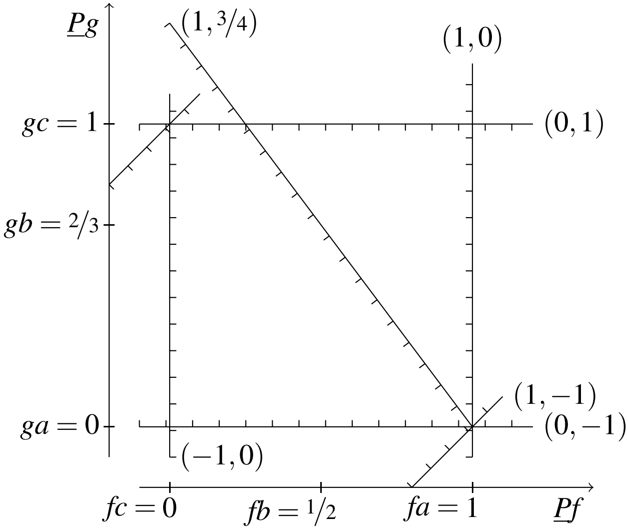
$\underline{P}g$
$(1,0)$
$gc = 1$
$(0,1)$
$gb = 2/3$
$ga = 0$
$(0,-1)$
$(-1,0)$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$



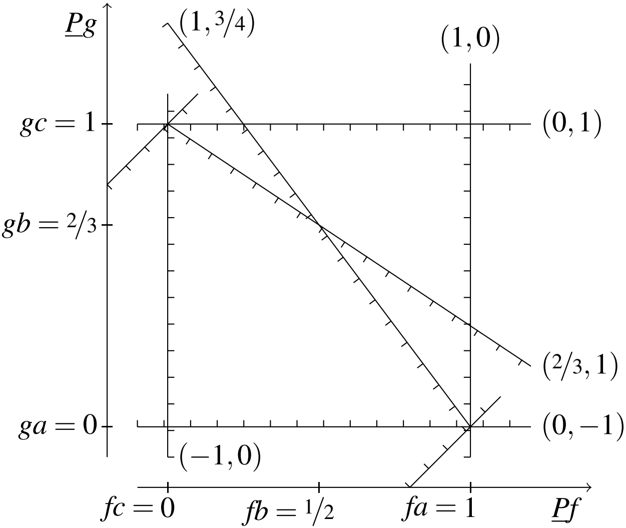
$\underline{P}g$
$(1,0)$
$(-1,1)$
$gc = 1$
$(0,1)$
$gb = 2/3$
$ga = 0$
$(0,-1)$
$(-1,0)$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$

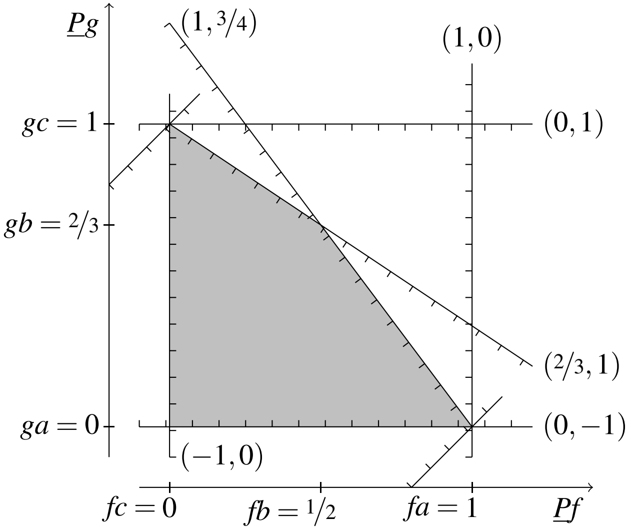


$\underline{P}g$
$(1,0)$
$(-1,1)$
$gc = 1$
$(0,1)$
$gb = 2/3$
$(1,-1)$
$ga = 0$
$(0,-1)$
$(-1,0)$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$

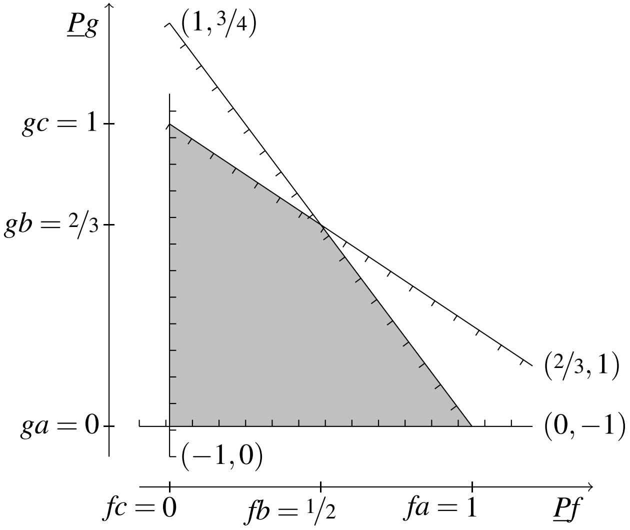


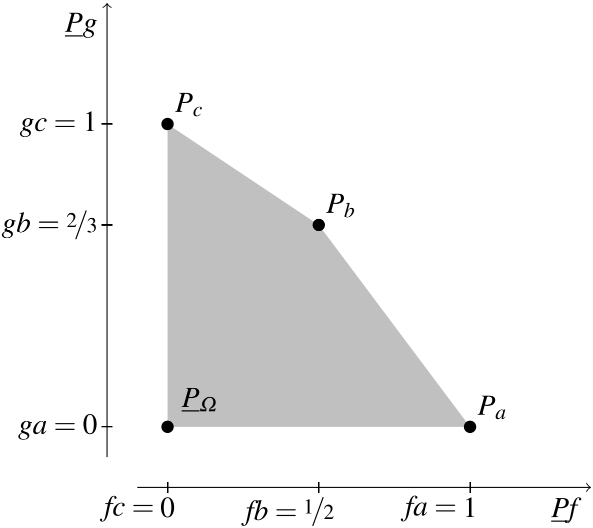
$\underline{P}g$
$(1, 3/4)$
$(1,0)$
$gc = 1$
$(0,1)$
$gb = 2/3$
$(1,-1)$
$ga = 0$
$(0,-1)$
$(-1,0)$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$


$\underline{P}g$
$(1, 3/4)$
$(1, 0)$
$gc = 1$
$(0, 1)$
$gb = 2/3$
$(2/3, 1)$
$ga = 0$
$(0, -1)$
$(-1, 0)$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$

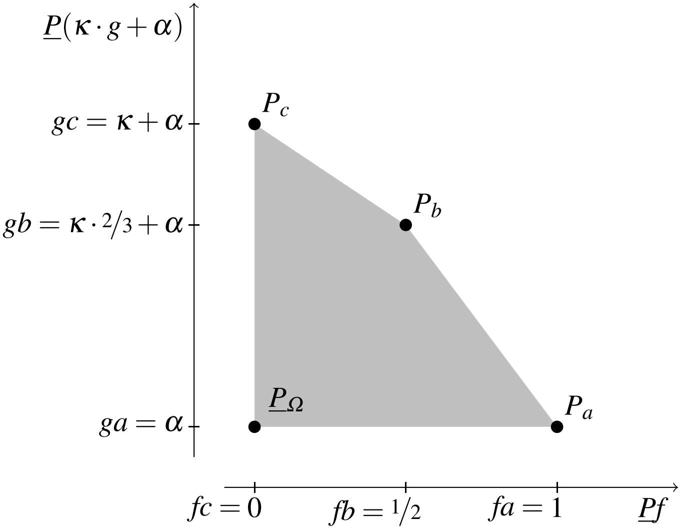


$\underline{P}g$
$(1, 3/4)$
$(1,0)$
$gc = 1$
$(0,1)$
$gb = 2/3$
$(2/3, 1)$
$ga = 0$
$(0,-1)$
$(-1,0)$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$

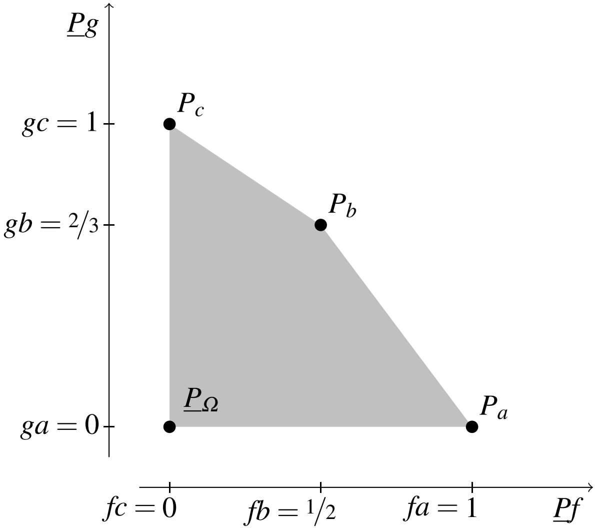

$\underline{P}g$
$(1, 3/4)$
$gc = 1$
$gb = 2/3$
$(2/3, 1)$
$ga = 0$
$(0, -1)$
$(-1, 0)$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$



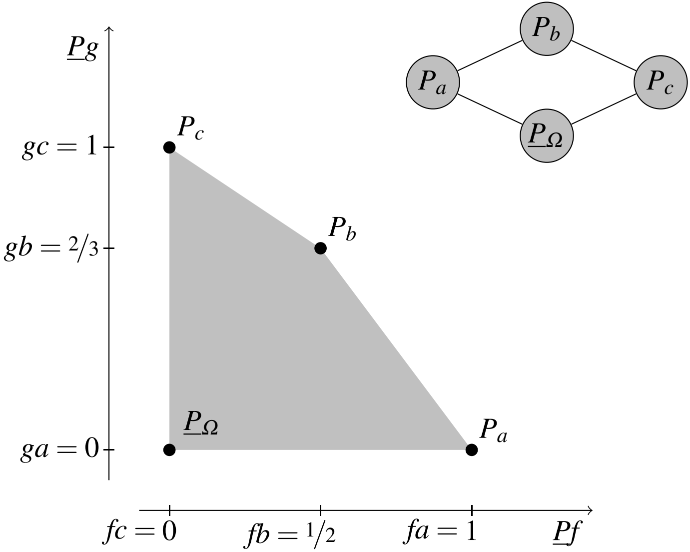
P̲g
P_c
gc = 1
P_b
gb = 2/3
P̲_Ω
P_a
ga = 0
fc = 0
fb = 1/2
fa = 1
P̲f



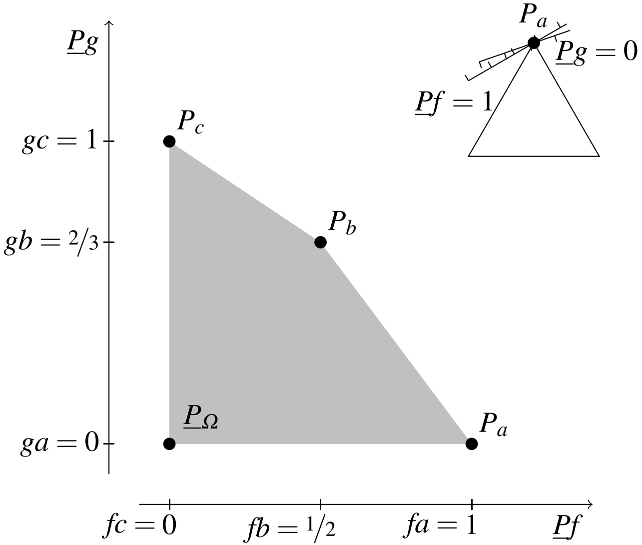
$\underline{P}(\kappa \cdot g + \alpha) = \kappa \cdot \underline{P}g + \alpha$ for all $\kappa \in \mathbb{R}_{\geq 0}$ and $\alpha \in \mathbb{R}$



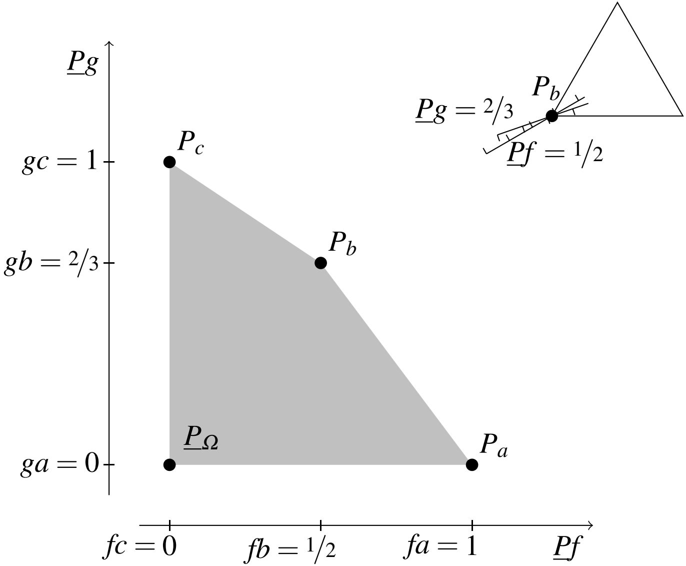
$\underline{P}g$
P_c
$gc = 1$
P_b
$gb = 2/3$
$\underline{P}_\Omega$
P_a
$ga = 0$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$



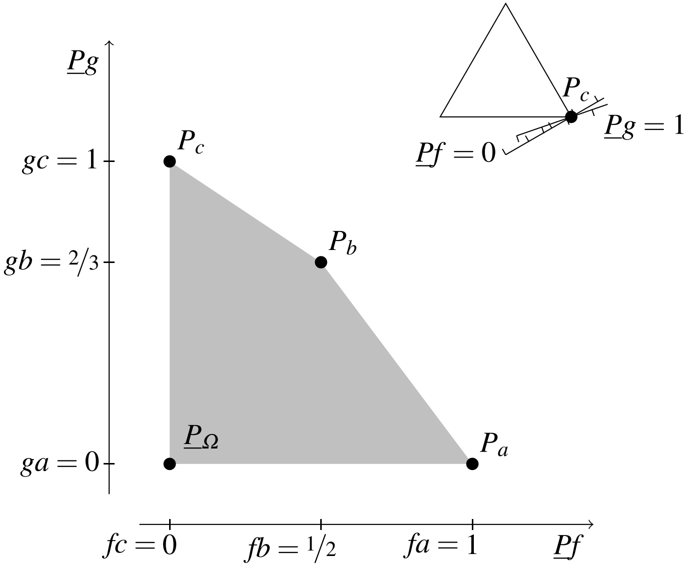
$\underline{P}g$
$gc = 1$
$gb = 2/3$
$ga = 0$
P_c
P_b
$\underline{P}_\Omega$
P_a
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$
P_b
P_a
P_c
$\underline{P}_\Omega$



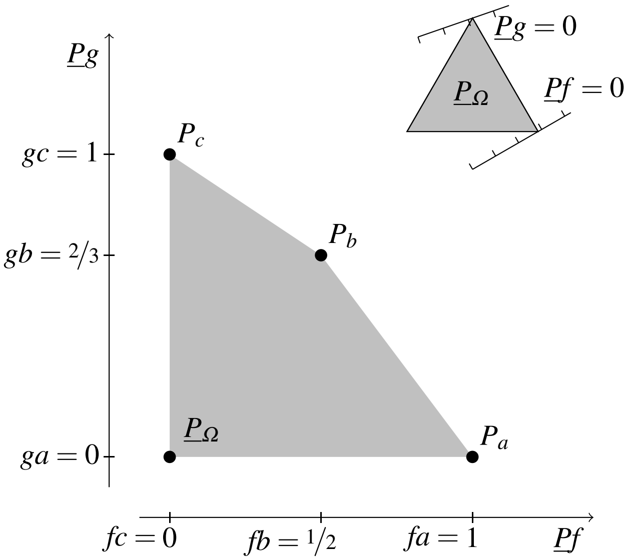
$\underline{P}g$
$gc = 1$
$gb = 2/3$
$ga = 0$
P_c
P_b
$\underline{P}_\Omega$
P_a
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$
P_a
$\underline{P}g = 0$
$\underline{P}f = 1$



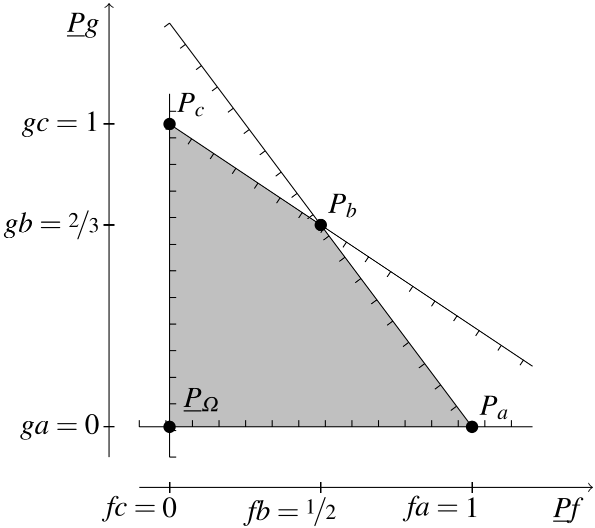
$\underline{P}g$
P_b
$\underline{P}g = 2/3$
$\underline{P}f = 1/2$
P_c
$gc = 1$
P_b
$gb = 2/3$
$\underline{P}_\Omega$
P_a
$ga = 0$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$



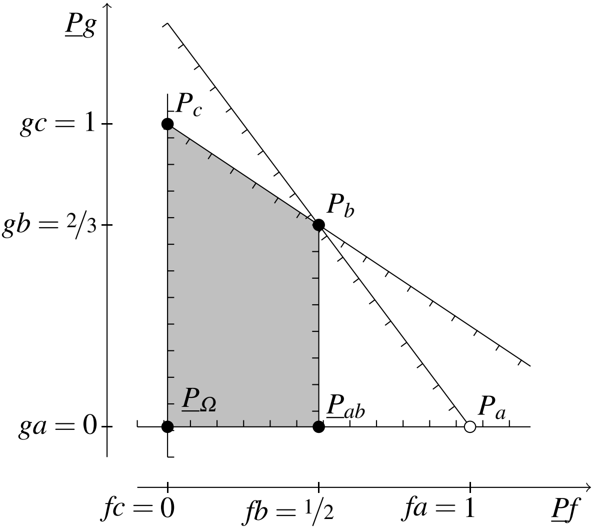
$\underline{P}g$
$gc = 1$
$gb = 2/3$
$ga = 0$
P_c
P_b
$\underline{P}_\Omega$
P_a
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$
P_c
$\underline{P}g = 1$
$\underline{P}f = 0$



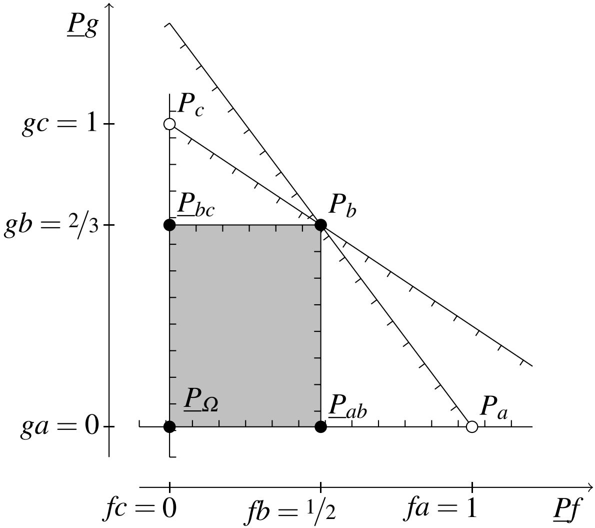
$\underline{P}g = 0$
$\underline{P}f = 0$
$\underline{P}_\Omega$
$\underline{P}g$
P_c
$gc = 1$
P_b
$gb = 2/3$
$\underline{P}_\Omega$
P_a
$ga = 0$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$



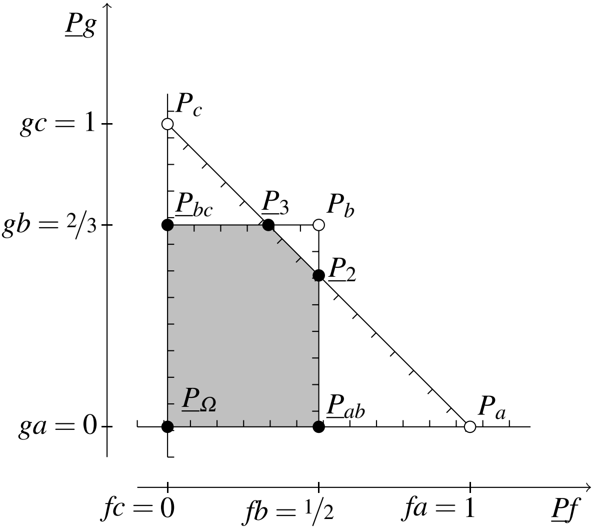
$\underline{P}g$
P_c
$gc = 1$
P_b
$gb = 2/3$
$\underline{P}_\Omega$
P_a
$ga = 0$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$



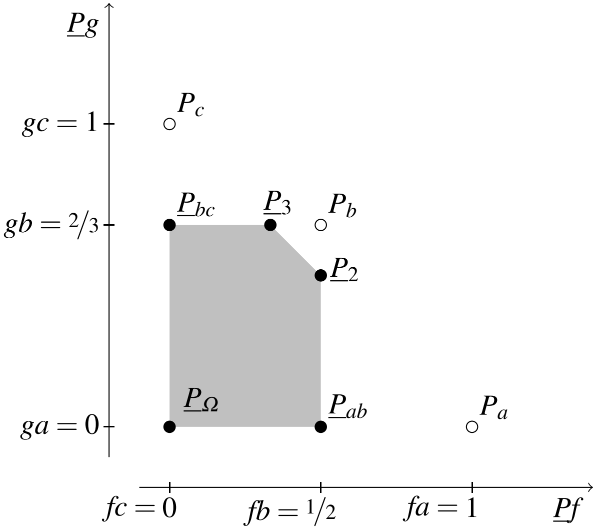
add I_a to $\mathscr{K}$



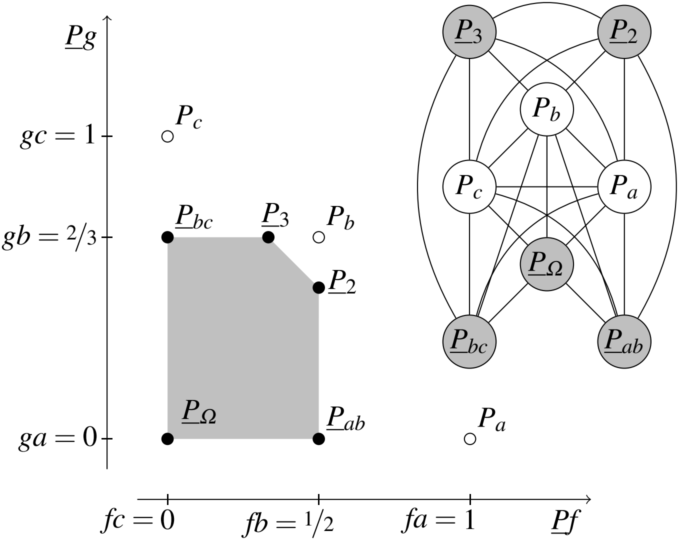
add I_c to $\mathscr{K}$



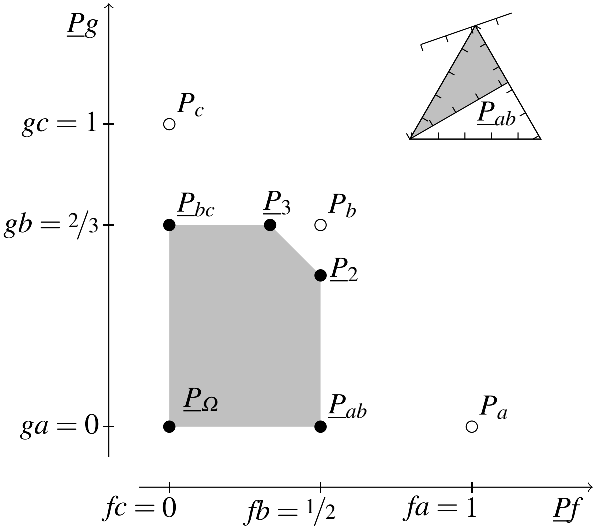
add I_b to $\mathscr{K}$



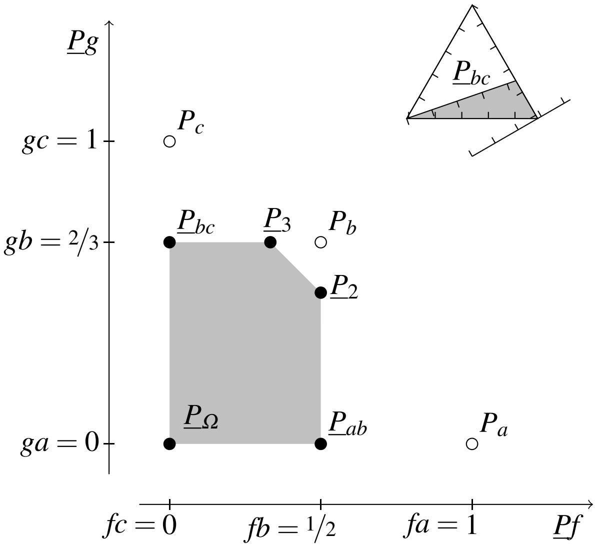
P̲g
Pc
gc = 1
P̲bc
P̲3
Pb
gb = 2/3
P̲2
P̲Ω
P̲ab
Pa
ga = 0
fc = 0
fb = 1/2
fa = 1
P̲f



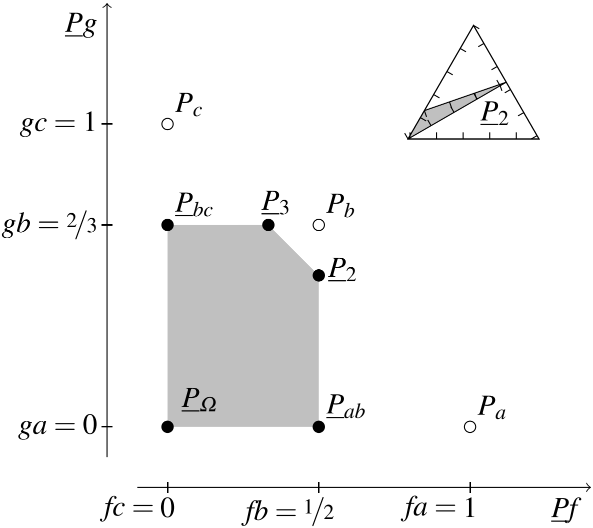
$\underline{P}g$
$gc = 1$
$gb = 2/3$
$ga = 0$
P_c
$\underline{P}_{bc}$
$\underline{P}_3$
P_b
$\underline{P}_2$
$\underline{P}_\Omega$
$\underline{P}_{ab}$
P_a
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$
$\underline{P}_3$
$\underline{P}_2$
P_b
P_c
P_a
$\underline{P}_\Omega$
$\underline{P}_{bc}$
$\underline{P}_{ab}$



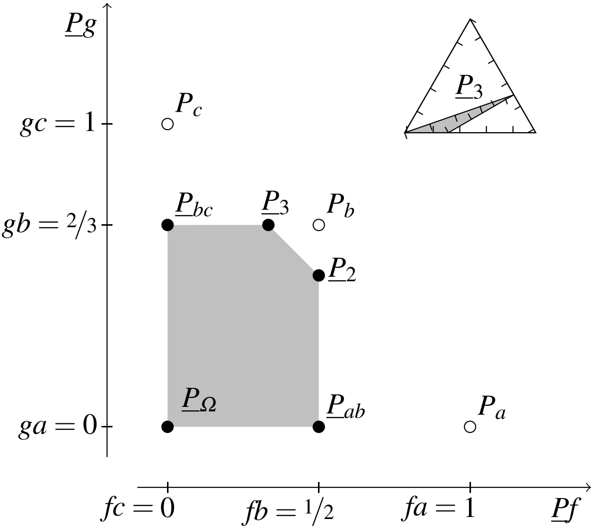
$\underline{P}g$
P_c
$gc = 1$
$\underline{P}_{ab}$
$\underline{P}_{bc}$
$\underline{P}_3$
P_b
$gb = 2/3$
$\underline{P}_2$
$\underline{P}_\Omega$
$\underline{P}_{ab}$
P_a
$ga = 0$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$



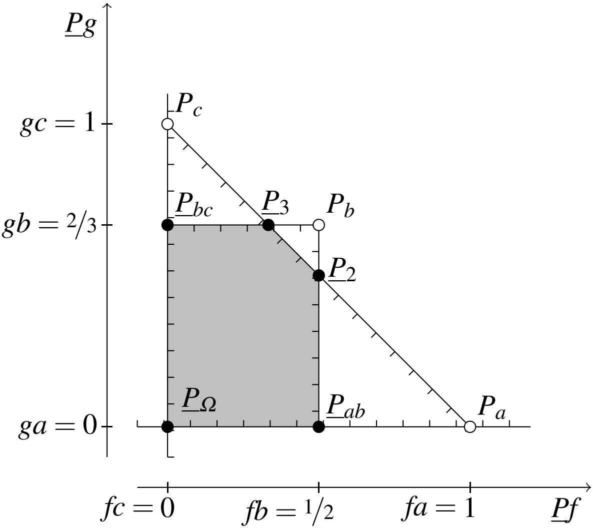
$\underline{P}g$
P_c
$gc = 1$
$\underline{P}_{bc}$
$\underline{P}_3$
P_b
$gb = 2/3$
$\underline{P}_2$
$\underline{P}_\Omega$
$\underline{P}_{ab}$
P_a
$ga = 0$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$
$\underline{P}_{bc}$



$\underline{P}g$
P_c
$gc = 1$
$\underline{P}_2$
$\underline{P}_{bc}$
$\underline{P}_3$
P_b
$gb = 2/3$
$\underline{P}_2$
$\underline{P}_\Omega$
$\underline{P}_{ab}$
P_a
$ga = 0$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$

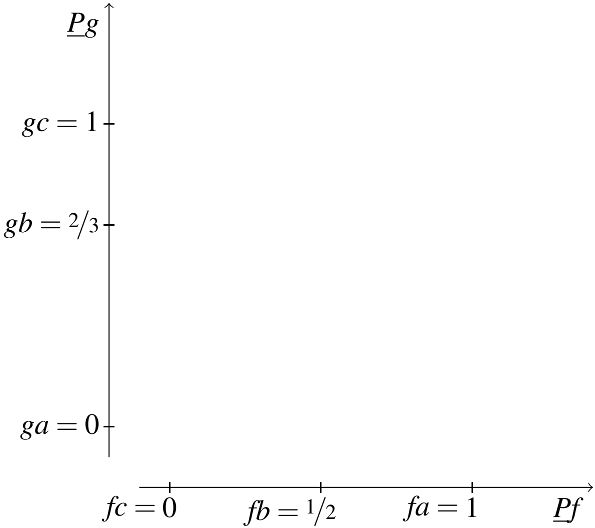


$\underline{P}g$
$gc = 1$
$gb = 2/3$
$ga = 0$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$
P_c
P_b
P_a
$\underline{P}_{bc}$
$\underline{P}_3$
$\underline{P}_2$
$\underline{P}_\Omega$
$\underline{P}_{ab}$
$\underline{P}_3$

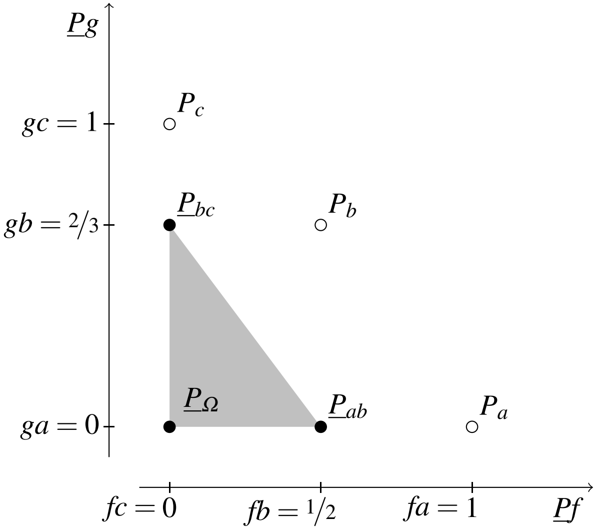


$\underline{P}g$
P_c
$gc = 1$
$\underline{P}_{bc}$
$\underline{P}_3$
P_b
$gb = 2/3$
$\underline{P}_2$
$\underline{P}_\Omega$
$\underline{P}_{ab}$
P_a
$ga = 0$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$

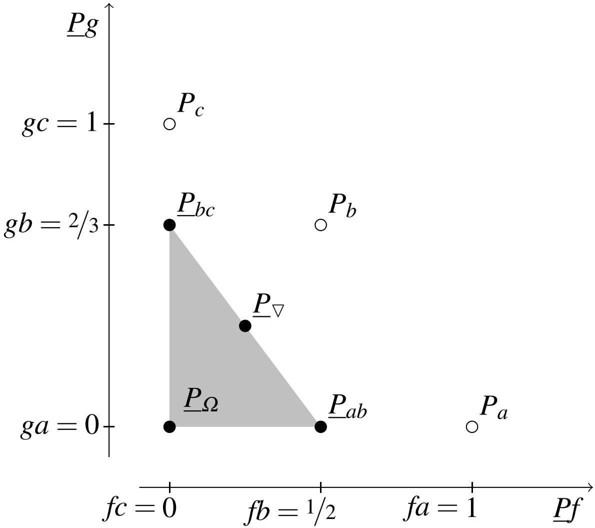
$\underline{P}$ on a lattice $\mathscr{K}$ is n-monotone iff $\underline{P}$ is monotone and $\underline{P}(\bigvee \hat{\mathscr{K}}) \geq \sum_{\check{\mathscr{K}} \subseteq \hat{\mathscr{K}}} (-1)^{|\check{\mathscr{K}}|+1} \cdot \underline{P}(\bigwedge \check{\mathscr{K}})$ for all $1 < k \leq n$ and $\hat{\mathscr{K}} \subseteq \mathscr{K}$



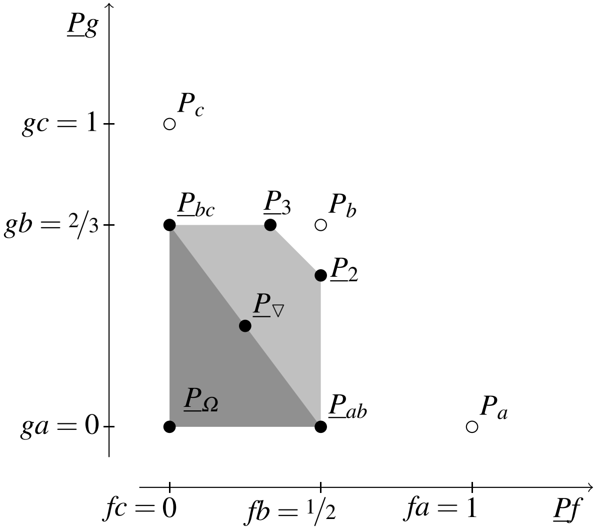
$\mathscr{K}$ lattice based on $\{f, g, I_a, I_b, I_c\}$, then project back on $\mathbb{R}^{\{f,g,I_a,I_b,I_c\}}$



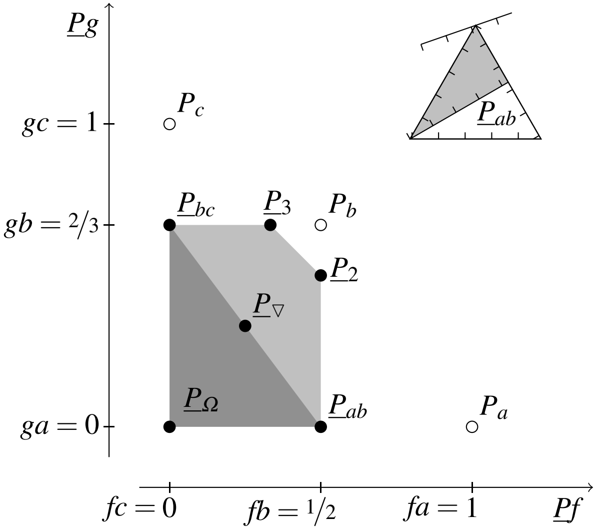
complete-monotonicity



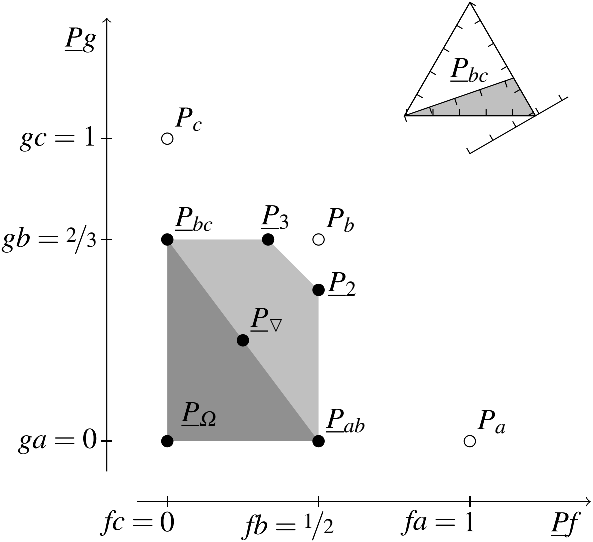
2-monotonicity



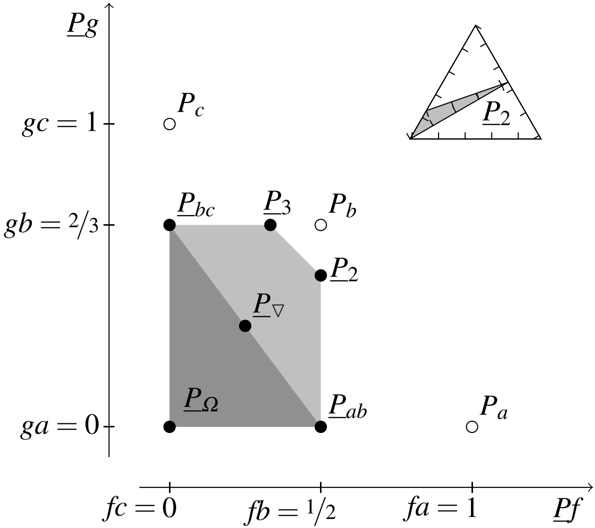
$\underline{P}g$
P_c
$gc = 1$
$\underline{P}_{bc}$
$\underline{P}_3$
P_b
$gb = 2/3$
$\underline{P}_2$
$\underline{P}_\nabla$
$\underline{P}_\Omega$
$\underline{P}_{ab}$
P_a
$ga = 0$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$



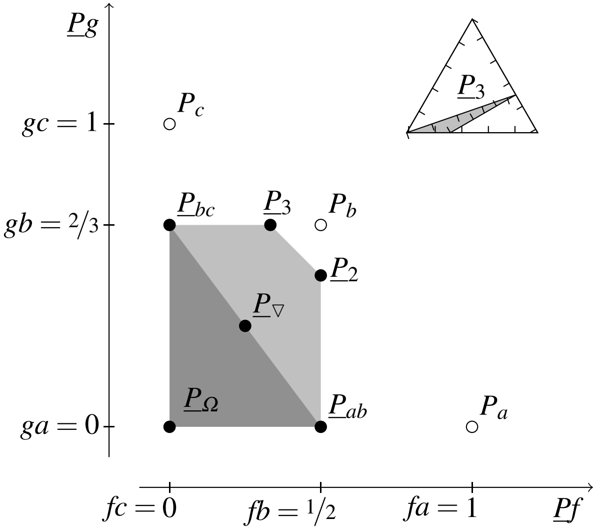
$\underline{P}g$
P_c
$gc = 1$
$\underline{P}_{ab}$
$\underline{P}_{bc}$
$\underline{P}_3$
P_b
$gb = 2/3$
$\underline{P}_2$
$\underline{P}_\nabla$
$\underline{P}_\Omega$
$\underline{P}_{ab}$
P_a
$ga = 0$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$



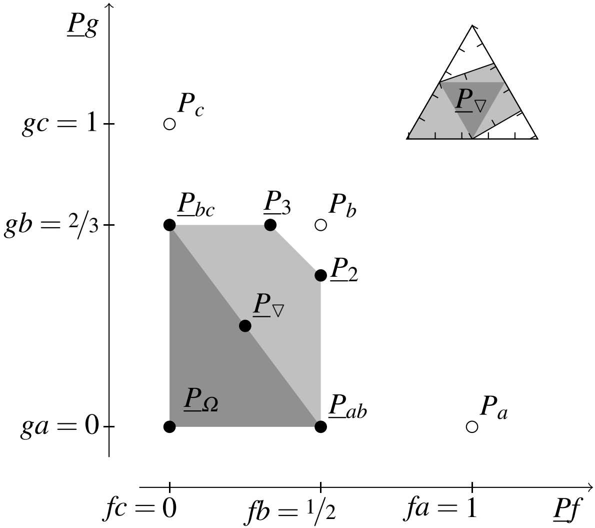
$\underline{P}g$
$gc = 1$
$gb = 2/3$
$ga = 0$
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$
P_c
P_b
P_a
$\underline{P}_{bc}$
$\underline{P}_3$
$\underline{P}_2$
$\underline{P}_\nabla$
$\underline{P}_\Omega$
$\underline{P}_{ab}$
$\underline{P}_{bc}$

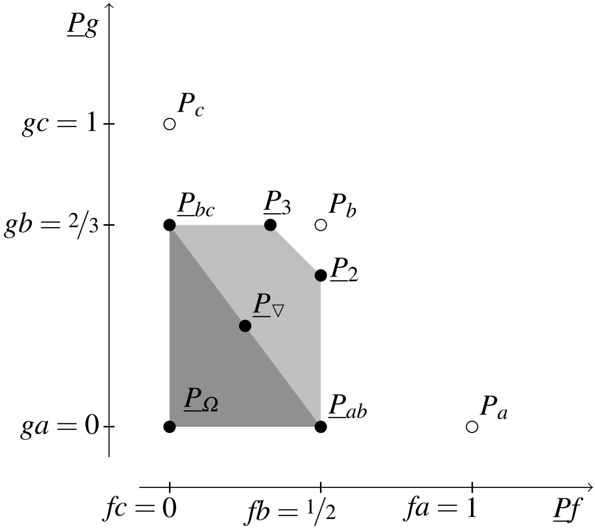


Pg
gc = 1
gb = 2/3
ga = 0
fc = 0
fb = 1/2
fa = 1
Pf
Pc
Pb
Pa
Pbc
P3
P2
P∇
PΩ
Pab
P2

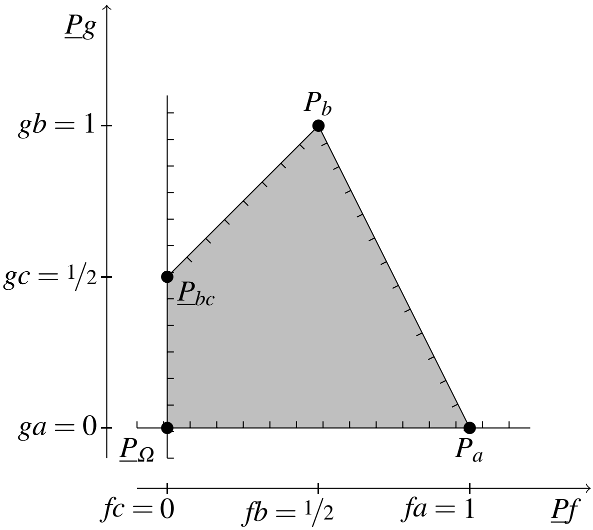


Pg
gc = 1
gb = 2/3
ga = 0
Pc
Pbc
P3
Pb
P2
P∇
PΩ
Pab
Pa
P3
fc = 0
fb = 1/2
fa = 1
Pf

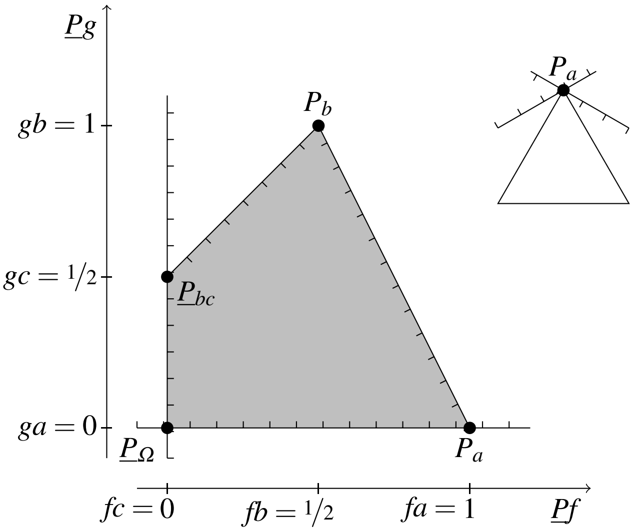

$\underline{P}g$
$gc = 1$
$gb = 2/3$
$ga = 0$
P_c
$\underline{P}_{bc}$
$\underline{P}_3$
P_b
$\underline{P}_2$
$\underline{P}_\nabla$
$\underline{P}_\Omega$
$\underline{P}_{ab}$
P_a
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$



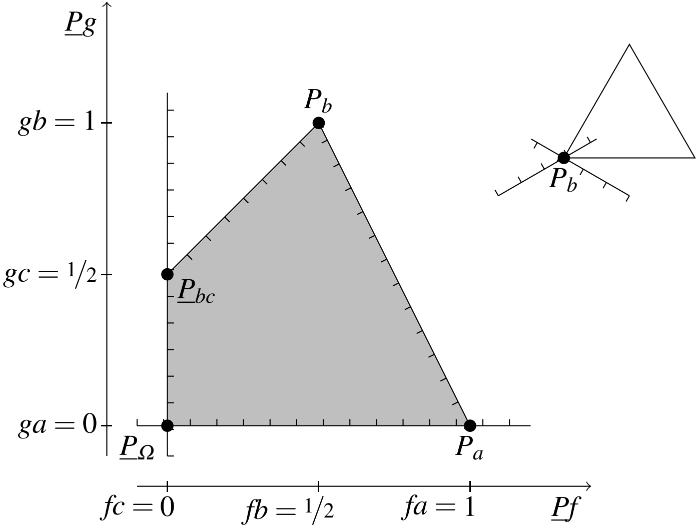
discouraging picture for n-monotone outer approximation accuracy



Pg
Pb
gb = 1
gc = 1/2
Pbc
ga = 0
PΩ
Pa
fc = 0
fb = 1/2
fa = 1
Pf



$\underline{P}g$
$gb = 1$
$gc = 1/2$
$ga = 0$
P_b
P_a
$\underline{P}_{bc}$
$\underline{P}_\Omega$
P_a
$fc = 0$
$fb = 1/2$
$fa = 1$
$\underline{P}f$



$\underline{P}g$
$gb = 1$
$gc = 1/2$
$ga = 0$
P_b
$\underline{P}_{bc}$
$\underline{P}_{\Omega}$
P_a
P_b
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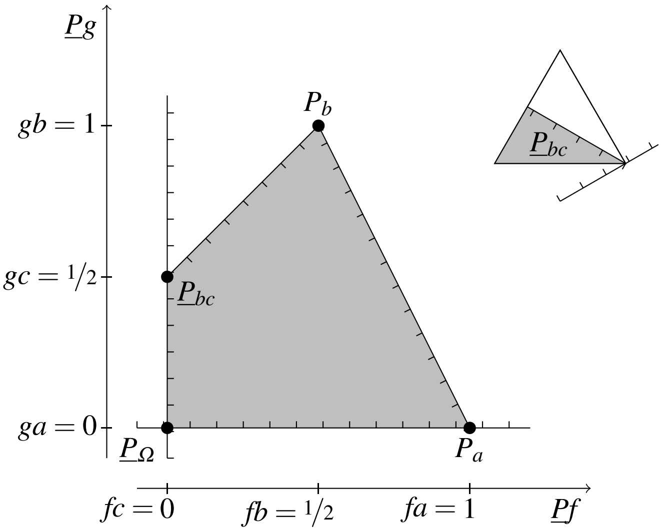

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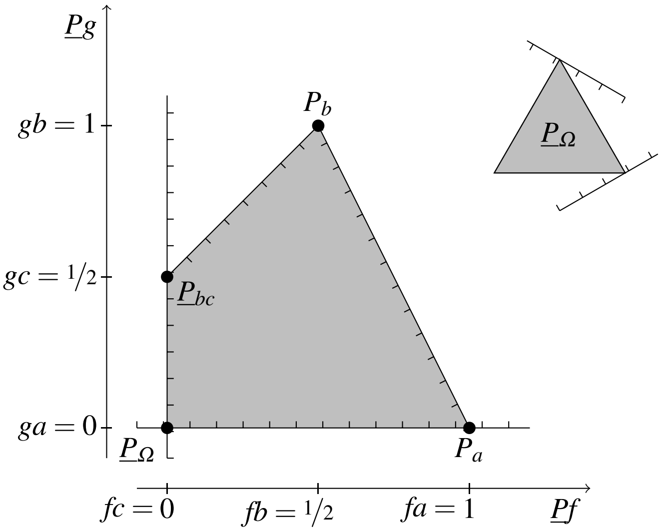

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P_b
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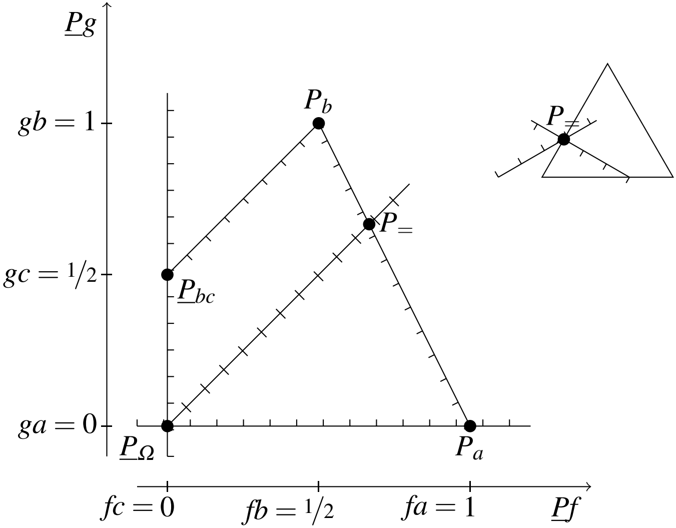
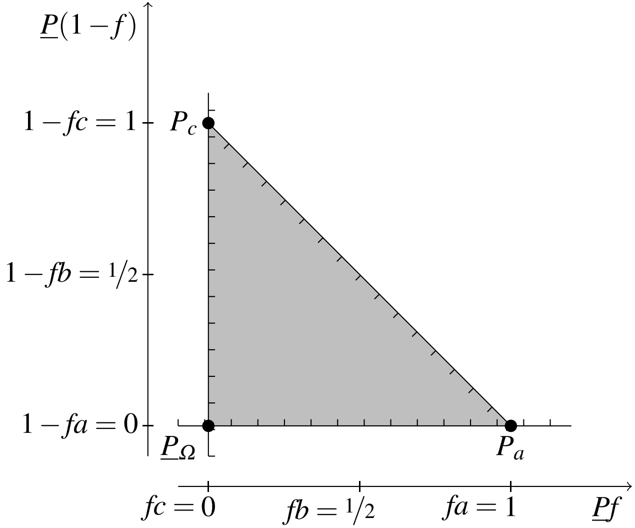
$\underline{P}f$
$\underline{P}_{\Omega}$

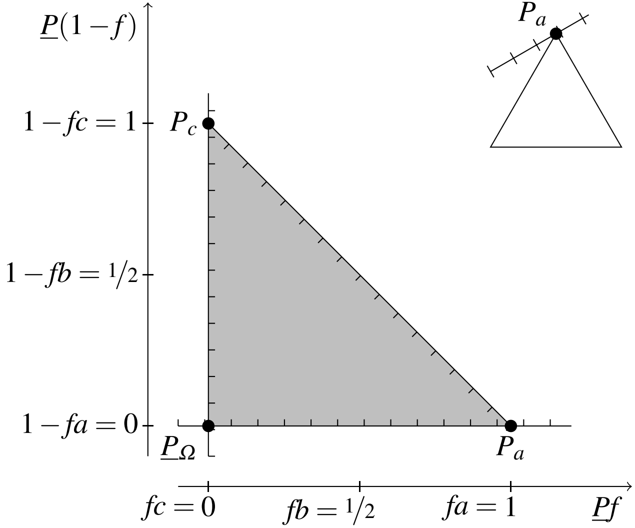


f and $1-f$ equal up to permutation; impose $\underline{P}g = \underline{P}f$

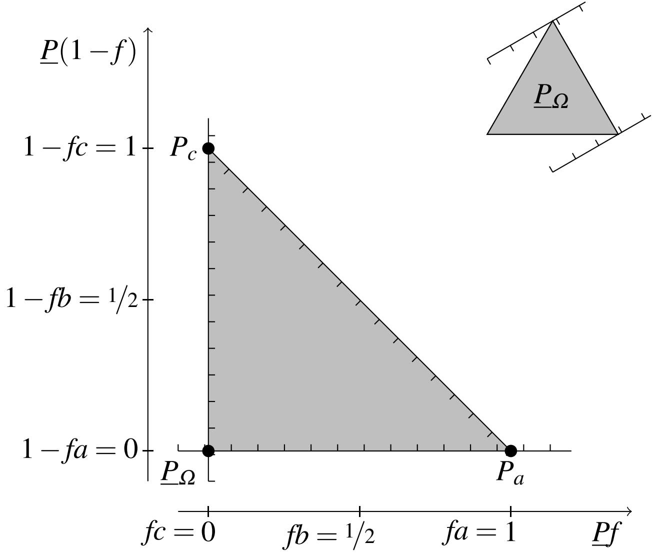
empty on purpose



$\underline{P}(1-f)$
$1-fc=1$
$1-fb=1/2$
$1-fa=0$
P_c
$\underline{P}_\Omega$
P_a
$fc=0$
$fb=1/2$
$fa=1$
$\underline{P}f$



$\underline{P}(1-f)$
P_a
$1-fc=1$
P_c
$1-fb={}^1/_2$
$1-fa=0$
$\underline{P}_\Omega$
P_a
$fc=0$
$fb={}^1/_2$
$fa=1$
$\underline{P}f$



$\underline{P}(1-f)$
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P_c
P_a
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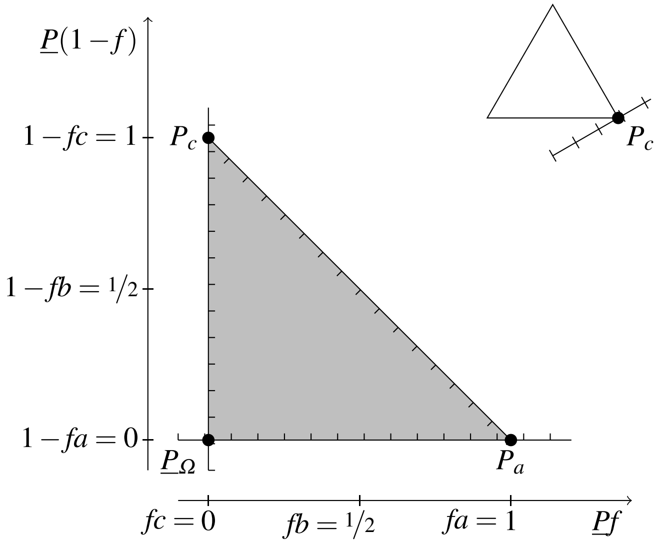
$fc=0$
$fb=1/2$
$fa=1$
$\underline{P}f$
$\underline{P}_\Omega$


$\underline{P}(1-f)$
$1-fc=1$
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P_c
$\underline{P}_\Omega$
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$fc=0$
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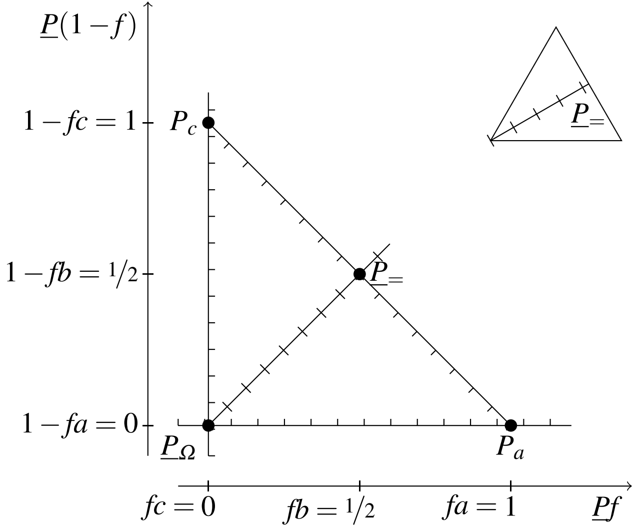

f and g equal up to permutation; impose $\underline{P}(1-f)=\underline{P}f$

Numbers, numbers, numbers

Combinatorics for coherent lower probabilities on different $\mathscr{K}$

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Combinatorics for coherent lower probabilities on different $\mathscr{K}$

Lower pmfs $|\mathscr{K}| = |\Omega|$ and $\#\lambda = \#\underline{P} = |\Omega| + 1$
(linear-vacuous)

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(linear-vacuous)

Upper pmfs $|\mathscr{K}| = |\Omega|$, $\#\lambda = 2 \cdot |\Omega| + 1$ and $\#\underline{P} = 2^{|\Omega|} + 1$
(*not* completely monotone)

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Upper pmfs $|\mathscr{K}| = |\Omega|$, $\#\lambda = 2 \cdot |\Omega| + 1$ and $\#\underline{P} = 2^{|\Omega|} + 1$
(*not* completely monotone)

Probability intervals $|\mathscr{K}| = 2 \cdot |\Omega|$ for $|\Omega| > 2$ and

$\lvert\Omega\rvert$	2	3	4	5	6	7	8	9	10
$\#\lambda$	3	9	16	20	24	28	32	36	40
$\#\underline{P}$	3	8	20	47	105	226	474	977	1991

(subset of 2-monotone)

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(subset of 2-monotone)

Lower probabilities $|\mathscr{K}| = 2^{|\Omega|}$ and

$\|\Omega\|$	2	3	4	5	6
$\#\lambda$	3 (3)	9 (17)	48 (179)	285 (7351)	? (?)
$\#\underline{P}$	3	8	402	?	?

More numbers, numbers, numbers

Combinatorics for n-monotone lower probabilities

More numbers, numbers, numbers

Combinatorics for n-monotone lower probabilities

Completely monotone $|\mathscr{K}| = 2^{|\Omega|}$, $\#\lambda = 2^{|\Omega|} + 3$ and $\#\underline{P} = 2^{|\Omega|} - 1$

More numbers, numbers, numbers

Combinatorics for n-monotone lower probabilities

Completely monotone $|\mathscr{K}| = 2^{|\Omega|}$, $\#\lambda = 2^{|\Omega|} + 3$ and $\#\underline{P} = 2^{|\Omega|} - 1$

2-monotone $|\mathscr{K}| = 2^{|\Omega|}$ and

$\lvert\Omega\rvert$	2	3	4	5	6
$\#\lambda$	7 (10)	13 (32)	32 (124)	89 (500)	? (?)
$\#\underline{P}$	3	8	41	117983	?

Still more numbers, numbers, numbers

Combinatorics for coherent lower previsions on different $\mathscr{K}$

Consider $\mathscr{K}$ consisting of gambles taking values in $\{\ell/k : 0 \leq \ell \leq k\}$

Still more numbers, numbers, numbers

Combinatorics for coherent lower previsions on different $\mathscr{K}$

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$|\Omega| = 3$ $|\mathscr{K}| = 2 \cdot k \cdot |\Omega|$, $\#\lambda = (2 \cdot k + 1) \cdot |\Omega|$, and
$\#\underline{P} = (3 \cdot k + 1) \cdot (3 \cdot k^2 - 4 \cdot k + 3)$

Still more numbers, numbers, numbers

Combinatorics for coherent lower previsions on different $\mathscr{K}$

Consider $\mathscr{K}$ consisting of gambles taking values in $\{\ell/k : 0 \leq \ell \leq k\}$

$|\Omega| = 3$ $|\mathscr{K}| = 2 \cdot k \cdot |\Omega|$, $\#\lambda = (2 \cdot k + 1) \cdot |\Omega|$, and $\#\underline{P} = (3 \cdot k + 1) \cdot (3 \cdot k^2 - 4 \cdot k + 3)$

$|\Omega| = 4$ computationally too demanding

Conclusion & To Do

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- We can get a view of (projected) sets of lower previsions for
  - building intuition;
  - pedagogical use;
  - more practical applications?

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  - building intuition;
  - pedagogical use;
  - more practical applications?
- For small $|\Omega|$ or $|\mathscr{K}|$ we can
  - efficiently check coherence for multiple lower previsions;
  - obtain the extreme lower previsions.
- Expand scope by adding contingent gambles?
  (for looking at independence concepts)