Inference & Desirability

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Context & assumptions

Possibility space $\mathcal{X}$ outcomes experiment

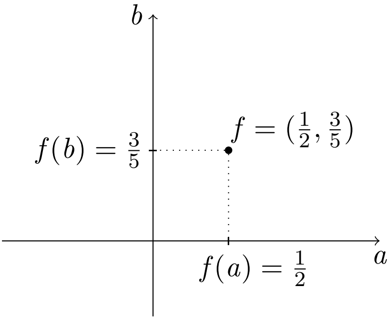
We—an intentional system uncertain about outcome experiment

Goal model our uncertainty/beliefs/information & use this model for reasoning

Gambles payoff depends on outcome,
bounded real-valued function on $\mathcal{X}$,
set of gambles $\mathcal{L}(\mathcal{X})$

Utility linear and precise

Gambles



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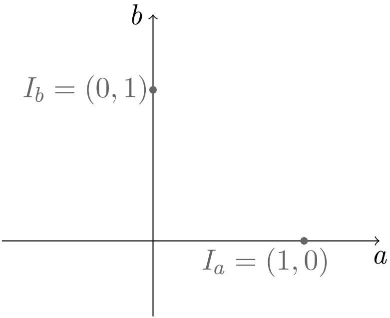

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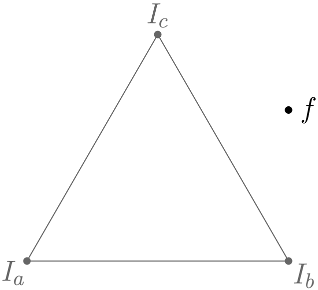

- $f = (-\frac{2}{3}, \frac{5}{6}, \frac{5}{6})$

Desirable gambles

Gamble f desirable when we accept the transaction

(i) the experiment's outcome x is determined

(ii) our capital is changed by $f(x)$

Our uncertainty model set of desirable gambles

Outline

Reasoning about and with sets of desirable gambles

- Rationality criteria
- Assessments avoiding partial (or sure) loss
- Coherent sets of desirable gambles
- Natural extension
- Desirability relative to subspaces with arbitrary vector orderings

Derived coherent sets of desirable gambles

Combining sets of desirable gambles

Partial preference orders

Maximally committal sets of strictly desirable gambles

Relationships with other, nonequivalent models

Constructive rationality criteria

It is reasonable to require that a set of desirable gambles $\mathcal{D} \subseteq \mathcal{L}(\mathcal{X})$ satisfies

$$\text{Positive scaling:} \quad \lambda > 0 \wedge f \in \mathcal{D} \Rightarrow \lambda f \in \mathcal{D}$$

$$\text{Addition:} \quad f, g \in \mathcal{D} \Rightarrow f + g \in \mathcal{D}$$

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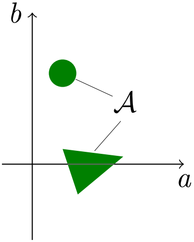
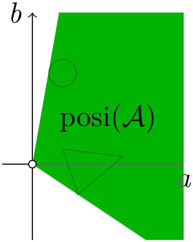
They extend an *assessment* $\mathcal{A} \subseteq \mathcal{L}(\mathcal{X})$ to

$$\mathrm{posi}(\mathcal{A}) := \left\{ \sum_{k=1}^{n} \lambda_k f_k \colon \lambda_k > 0 \wedge f_k \in \mathcal{L}(\mathcal{X}) \wedge n \in \mathbb{N} \right\}$$





Constraining rationality criteria

Comparing gambles the ordinary vector ordering is defined by

$$f \geq g \Leftrightarrow f - g \geq 0 \Leftrightarrow (f-g) \in \mathcal{L}_0^+(\mathcal{X}) \Leftrightarrow \inf(f-g) \geq 0$$

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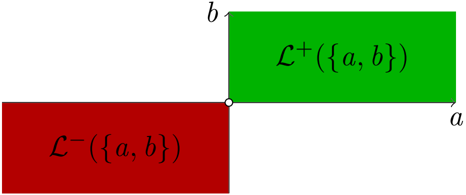

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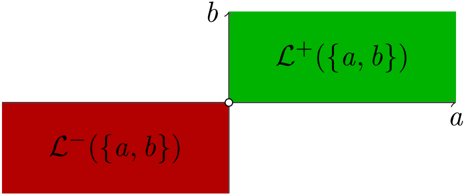

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Accepting partial gain: $f > 0 \Rightarrow f \in \mathcal{D}$

Avoiding partial loss: $f < 0 \Rightarrow f \notin \mathcal{D}$

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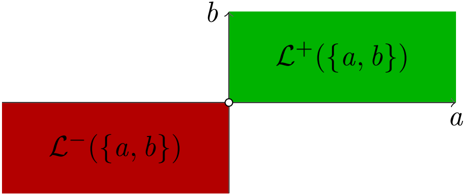

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Accepting partial gain:	$\mathcal{L}^+(\mathcal{X}) \subseteq \mathcal{D}$
Avoiding partial loss:	$\mathcal{D} \cap \mathcal{L}^-(\mathcal{X}) = \emptyset$

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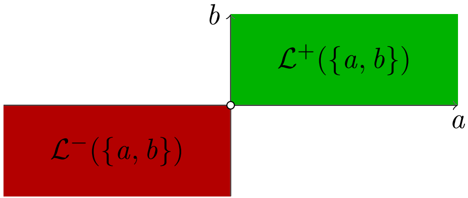

If $\mathcal{D} \subseteq \mathcal{L}(\mathcal{X})$ accepts partial gain and avoids partial loss, then it also satisfies

Accepting sure gain:	$\inf f > 0 \Rightarrow f \in \mathcal{D}$
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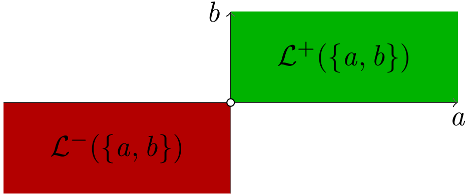

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Accepting sure gain:	$\operatorname{int}(\mathcal{L}^+(\mathcal{X})) \subseteq \mathcal{D}$
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Assessments & partial loss

An assessment $\mathcal{A} \subseteq \mathcal{L}(\mathcal{X})$ *avoids partial loss* iff

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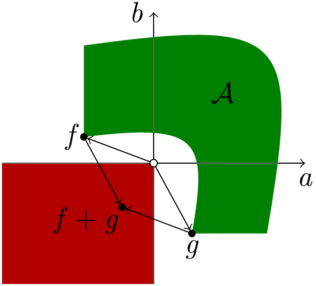
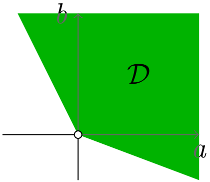


Coherent sets of desirable gambles

Coherence: A set of desirable gambles $\mathcal{D} \subseteq \mathcal{L}(\mathcal{X})$ is coherent if it satisfies all four rationality criteria.

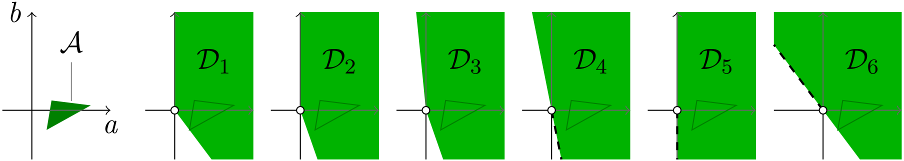
Geometry: It is a convex cone containing the positive orthant $\mathcal{L}^{+}(\mathcal{X})$, but excluding the negative orthant $\mathcal{L}^{-}(\mathcal{X})$.



Set of coherent sets $\mathbb{D}(\mathcal{X})$

Coherent extensions

Coherent extensions of an assessment $\mathcal{A} \subseteq \mathcal{L}(\mathcal{X})$ Any encompassing coherent set of desirable gambles



Set of coherent extensions $\mathbb{D}_{\mathcal{A}} \coloneqq \{\mathcal{D} \in \mathbb{D}(\mathcal{X}) \colon \mathcal{A} \subseteq \mathcal{D}\}$

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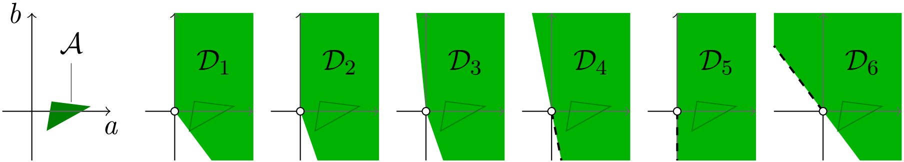

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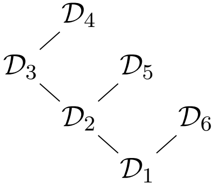
Inclusion based partial order of extensions that are more/less *committal*



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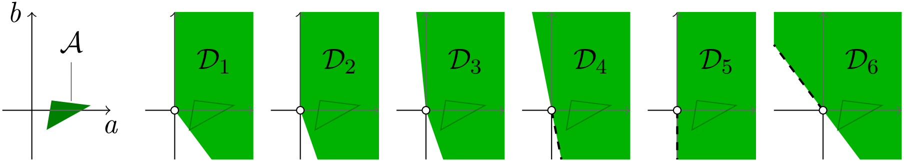

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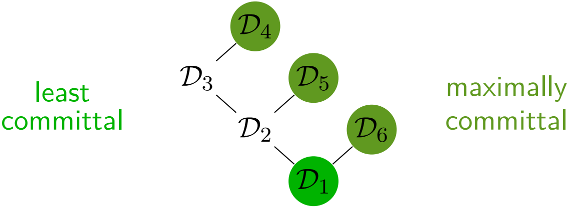
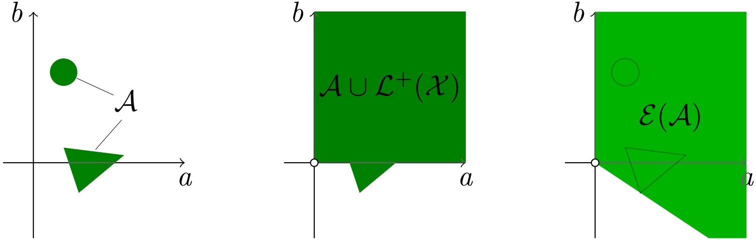


Natural extension

Given the constructive rationality criteria and accepting partial gains, there is a *natural extension* of an assessment $\mathcal{A} \subseteq \mathcal{L}(\mathcal{X})$:

$$\begin{aligned}\mathcal{E}(\mathcal{A}) &\coloneqq \operatorname{posi}(\mathcal{A} \cup \mathcal{L}^+(\mathcal{X})) \\ &= \operatorname{posi}(\mathcal{A}) \cup \mathcal{L}^+(\mathcal{X}) \cup \big(\operatorname{posi}(\mathcal{A}) + \mathcal{L}^+(\mathcal{X})\big)\end{aligned}$$



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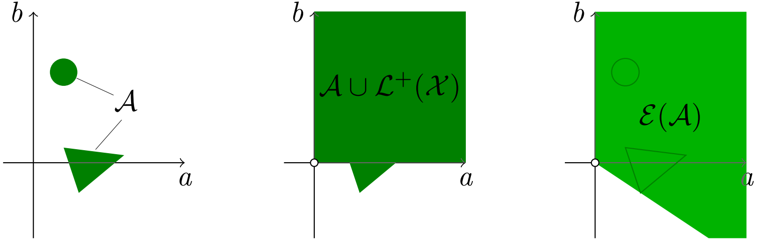

Natural Extension Theorem

The natural extension $\mathcal{E}(\mathcal{A})$ of $\mathcal{A} \subseteq \mathcal{L}(\mathcal{X})$ coincides with its least committal coherent extension $\bigcap \mathbb{D}_{\mathcal{A}}$ if and only if $\mathcal{A}$ avoids partial loss.

Natural extension is the prime tool for *deductive inference* in desirability.

Desirability relative to subspaces with arbitrary vector orderings

Desirability up until now ‘relative’ to $\mathcal{L}(\mathcal{X})$, the linear space of all gambles on $\mathcal{X}$, with the ordinary vector ordering determined by $\mathcal{L}^+(\mathcal{X})$ and $\mathcal{L}_0^+(\mathcal{X}) = \mathcal{L}^+(\mathcal{X}) \cup \{0\}$

Desirability relative to a linear subspace $\mathcal{K}$ of $\mathcal{L}(\mathcal{X})$

Arbitrary vector ordering determined by cones $\mathcal{C} \subset \mathcal{L}(\mathcal{X})$ and $\mathcal{C}_0 = \mathcal{C} \cup \{0\}$

Exercises I

1. Possibility space $\{a, b\}$; given are assessments

$$\begin{aligned}
\mathcal{A}_1 &\coloneqq \{(-1000, 1)\},\\
\mathcal{A}_2 &\coloneqq \{(-1000, 0), (\tfrac{1}{4}, \tfrac{1}{2}), (6, 3)\},\\
\mathcal{A}_3 &\coloneqq \{(-1000, 1), (\tfrac{1}{4}, -\tfrac{1}{2})\},\\
\mathcal{A}_4 &\coloneqq \{(-1, 2), (\tfrac{1}{2}, -\tfrac{1}{4})\}.
\end{aligned}$$

   1.1 Does $\mathcal{A}_i$ avoid sure loss?
   1.2 Does $\mathcal{A}_i$ avoid partial loss?
   1.3 Does $\operatorname{posi}(\mathcal{A}_i)$ accept sure gain?
   1.4 Does $\operatorname{posi}(\mathcal{A}_i)$ accept partial gain?
   1.5 If $\mathcal{A}_i$ avoids sure loss, describe $\mathcal{E}(\mathcal{A}_i)$ by giving its extreme rays (as sup-norm one vectors).
   1.6 Order all of the resulting $\mathcal{E}(\mathcal{A}_i)$ according to how committal they are.

Exercises II

2. Possibility space $\{a, b, c\}$; given are assessments

$$\mathcal{A}_5 := \{(1, -2, 0), (0, 1, -2)\},$$
$$\mathcal{A}_6 := \{(1, -2, 0), (0, 2, -4), (-8, 0, 4)\},$$
$$\mathcal{A}_7 := \{(-1, 0, 4), 6I_b - 1\}.$$

   2.1 Repeat the subquestions of Exercise 1.
   2.2 Represent $\mathcal{E}(\mathcal{A}_7)$ in the sum-one plane of $\mathcal{L}(\{a, b, c\})$.

3. Repeat Exercise 1 for vector orderings defined by the cones.

$$\mathcal{C}_1 := \operatorname{posi}(\{(1, \tfrac{1}{10}), (0, 1)\}),$$
$$\mathcal{C}_2 := \operatorname{posi}(\{(1, -\tfrac{1}{10}), (0, 1)\}),$$
$$\mathcal{C}_3 := \operatorname{posi}(\{(1, -\tfrac{1}{10}), (0, -1)\}).$$

4. Prove the Natural Extension Theorem.

Outline

Reasoning about and with sets of desirable gambles

Derived coherent sets of desirable gambles

- Gamble space transformations
- Conditional sets of desirable gambles
- Marginal sets of desirable gambles

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Relationships with other, nonequivalent models

Gamble space transformations that preserve coherence

Possibility spaces $\mathcal{X}$ and $\mathcal{Z}$

Transformation Γ from $\mathcal{L}(\mathcal{Z})$ to $\mathcal{L}(\mathcal{X})$

Conditions for preserving coherence

Positive homogeneity:	$\lambda > 0 \Rightarrow \Gamma(\lambda f) = \lambda \Gamma f$
Additivity:	$\Gamma(f + g) = \Gamma f + \Gamma g$
Positivity:	$f > 0 \Leftrightarrow \Gamma f > 0$
Negativity:	$f < 0 \Leftrightarrow \Gamma f < 0$

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which imply

Linearity:	$\lambda \in \mathbb{R} \Rightarrow \Gamma(\lambda f + g) = \lambda \Gamma f + \Gamma g$
Monotonicity:	$f > g \Leftrightarrow \Gamma f > \Gamma g$

Coherence Preserving Transformation Proposition

A transformation preserves coherence if and only if it is linear and monotone.

Transformation of a set of desirable gambles

$$\mathcal{D}_{\Gamma} \coloneqq \{h \in \mathcal{L}(\mathcal{Z})\colon \Gamma h \in \mathcal{D}\}$$

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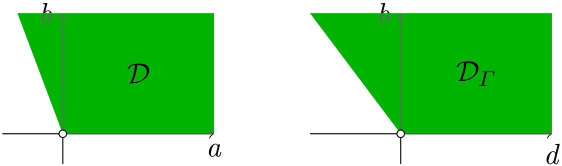
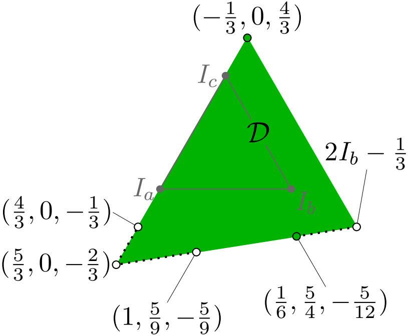


- $\Gamma : \mathcal{L}(\{d, b\}) \to \mathcal{L}(\{a, b\})$
- $(\Gamma h)(a) = \frac{1}{2} h(d)$ and $(\Gamma h)(b) = h(b)$

Taking a slice of a set of desirable gambles



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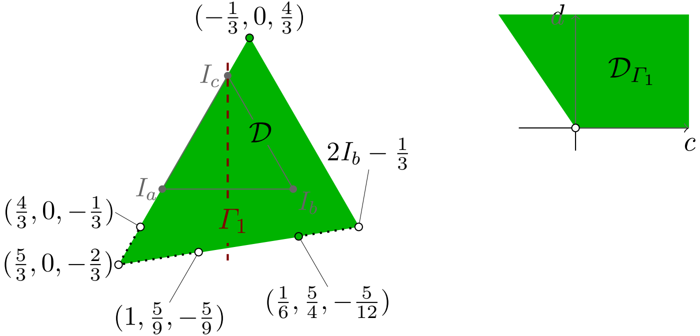

- $\Gamma_1 : \mathcal{L}(\{c, d\}) \to \mathcal{L}(\{a, b, c\})$
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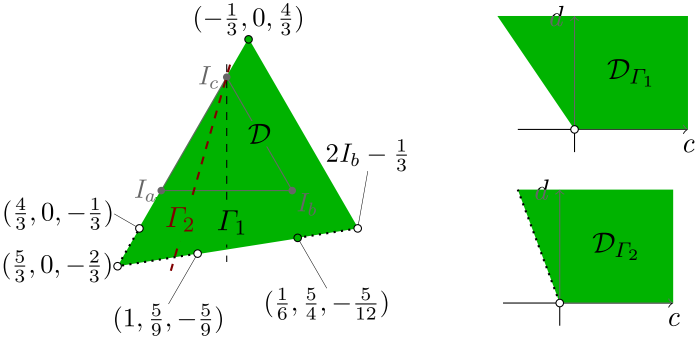

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- $\Gamma_2 : \mathcal{L}(\{c, d\}) \to \mathcal{L}(\{a, b, c\})$
- $(\Gamma_2 h)(a) = \frac{3}{4} h(d)$, $(\Gamma_2 h)(b) = \frac{1}{4} h(d)$ and $(\Gamma_2 h)(c) = h(c)$

Conditional sets of desirable gambles

Conditioning event $B \subseteq \mathcal{X}$ is what the experiment's outcome is assumed to belong to

Contingent gambles are those for which, if B does not occur, status quo is maintained

Transformation $\dagger_{B^c}$ maps gambles on B to contingent gambles on $\mathcal{X}$:

$$(\dagger_{B^c} h)(x) = \begin{cases} h(x), & x \in B, \\ 0, & x \in B^c, \end{cases}$$

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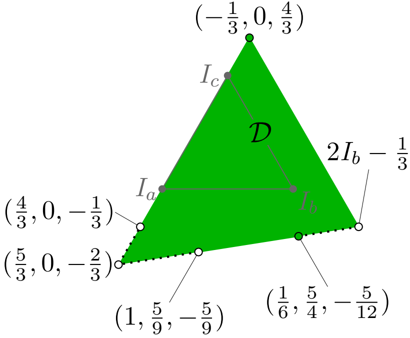
Conditional set of desirable gambles Given a set of desirable gambles $\mathcal{D} \subseteq \mathcal{L}(\mathcal{X})$, the set of desirable gambles conditional on B is

$$\mathcal{D}|B \coloneqq \mathcal{D}_{\nmid_{B^c}} = \{h \in \mathcal{L}(B) \colon \nmid_{B^c} h \in \mathcal{D}\}$$

- Other formats: $\nmid_{B^c}(\mathcal{D}|B) = \{f \in \mathcal{D} \colon f = fI_B\}$ and $\nmid_{B^c}(\mathcal{D}|B) + \nmid_B(\mathcal{L}(B^c)) = \{f \in \mathcal{L}(\mathcal{X}) \colon fI_B \in \mathcal{D}\}$
- Can be used as an *updated* set of desirable gambles

Conditional sets of desirable gambles: example



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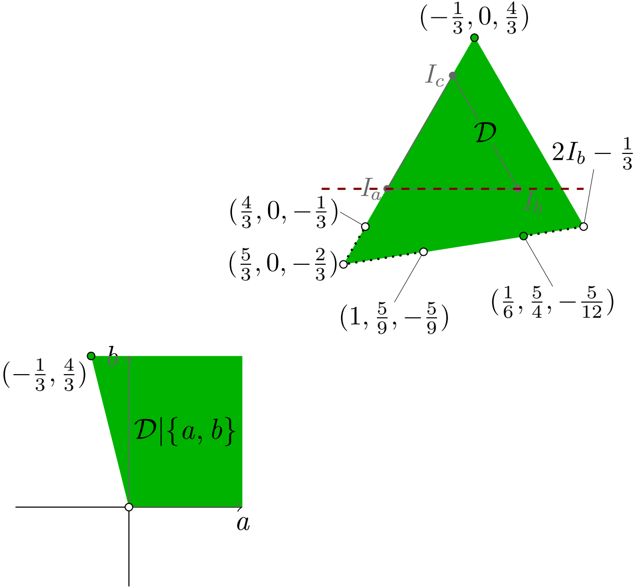

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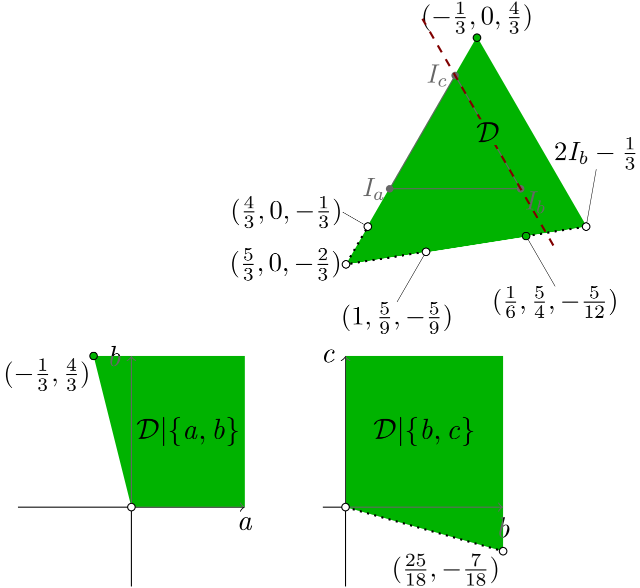

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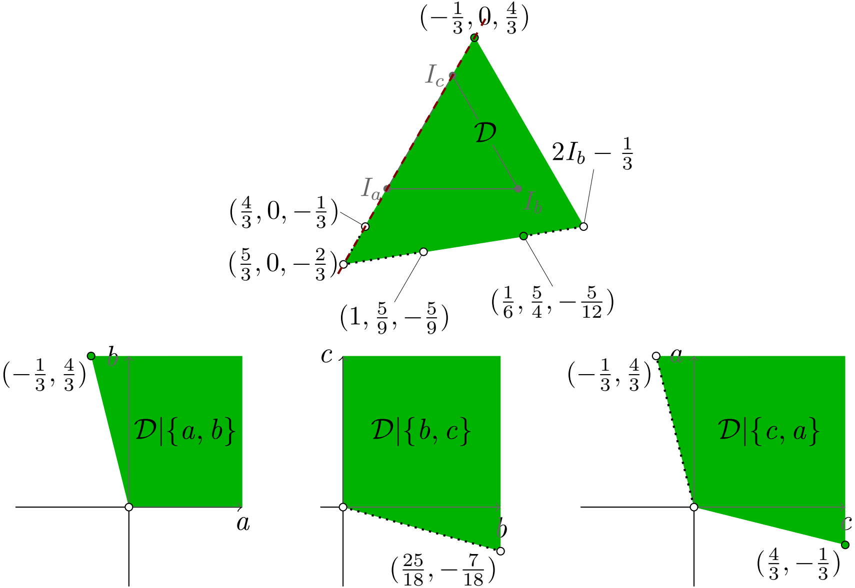

Marginal sets of desirable gambles

Cartesian product possibility space $\mathcal{X} = \mathcal{Y} \times \mathcal{Z}$, focus on $\mathcal{Y}$-component (ignore $\mathcal{Z}$-component)

Cylindrical extension $\uparrow_{\mathcal{Z}}$ maps gambles from the source gamble space to its cartesian product with $\mathcal{L}(\mathcal{Z})$:

$$(\uparrow_{\mathcal{Z}} h)(y, z) = h(y)$$

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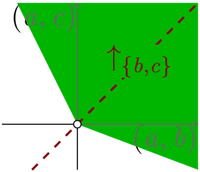
Marginal set of desirable gambles Given a set of desirable gambles $\mathcal{D} \subseteq \mathcal{L}(\mathcal{Y} \times \mathcal{Z})$, its $\mathcal{Y}$-*marginal* is

$$\mathcal{D}{\downarrow}\mathcal{Y} \coloneqq \mathcal{D}_{\uparrow_{\mathcal{Z}}} = \{h \in \mathcal{L}(\mathcal{Y}) \colon \uparrow_{\mathcal{Z}} h \in \mathcal{D}\}$$



Marginals for surjective maps and partitions

Essential features of marginalization:

Surjective map $\gamma_{\downarrow\mathcal{Y}}$ from $\mathcal{X} = \mathcal{Y} \times \mathcal{Z}$ to $\mathcal{Y}$ such that $\uparrow_{\mathcal{Z}} h = h \circ \gamma_{\downarrow\mathcal{Y}}$:

$$\gamma_{\downarrow\mathcal{Y}}(y, z) = y$$

Partition $\mathcal{B}_{\gamma_{\downarrow\mathcal{Y}}}$ can function as the possibility space of the $\mathcal{Y}$-marginal:

$$\mathcal{B}_{\gamma_{\downarrow\mathcal{Y}}} := \{\gamma_{\downarrow\mathcal{Y}}^{-1}(y) \colon y \in \mathcal{Y}\} = \{\{y\} \times \mathcal{Z} \colon y \in \mathcal{Y}\}$$

Marginals for surjective maps and partitions

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Generalization from the Cartesian product case:

Surjective map γ Associated transformation $\Gamma_\gamma h = h \circ \gamma$
and partition $\mathcal{B}_\gamma := \{\gamma^{-1}(y) \colon y \in \mathcal{Y}\}$;
resulting γ-*marginal* $\mathcal{D}_\gamma := \mathcal{D}_{\Gamma_\gamma}$.

Partition $\mathcal{B}$ Analogous; define $\gamma_{\mathcal{B}}$ for all $x \in \mathcal{X}$ by
letting $\gamma_{\mathcal{B}}(x)$ equal that B in $\mathcal{B}$ for which $x \in B$.

Outline

Reasoning about and with sets of desirable gambles

Derived coherent sets of desirable gambles

Combining sets of desirable gambles

- Joining compatible individuals
- Marginal extension

Partial preference orders

Maximally committal sets of strictly desirable gambles

Relationships with other, nonequivalent models

Joining compatible individuals

How can we combine *individual* sets of desirable gambles into a *joint*?

- View the individual sets as derived from the joint:
  specify the transformations between the individual gamble spaces and the joint gamble space.
- The union of the transformed individual sets is taken as an assessment.
- Check whether this the individual sets are *compatible*;
  i.e., if the assessment avoids partial loss
- If so, the natural extension of the assessment is the joint;
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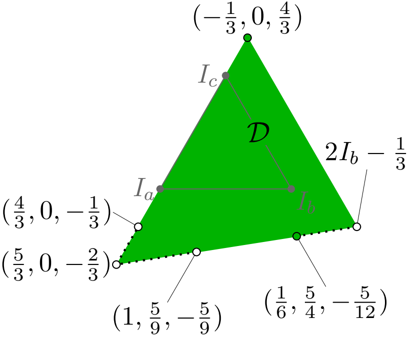
Consider the following individually coherent conditional sets of desirable gambles:

- $\mathcal{E}(\{(-2,1)\}) \subset \mathcal{L}(\{a,b\})$; a contingent gamble: $(-2,1,0)$
- $\mathcal{E}(\{(-2,1)\}) \subset \mathcal{L}(\{b,c\})$; a contingent gamble: $(0,-2,1)$
- $\mathcal{E}(\{(-2,1)\}) \subset \mathcal{L}(\{c,a\})$; a contingent gamble: $(1,0,-2)$

They are incompatible: the sum of the given contingent desirable gambles, $(-1,-1,-1)$, incurs sure loss.

Combining sets of desirable gambles: example



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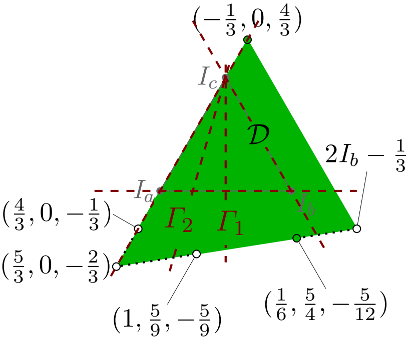

$$\begin{aligned}\mathcal{A} \coloneqq \Gamma_1(\mathcal{D}_{\Gamma_1}) \cup \Gamma_2(\mathcal{D}_{\Gamma_2}) &\cup \nmid_{\{c\}}(\mathcal{D}|\{a,b\}) \\ &\cup \nmid_{\{a\}}(\mathcal{D}|\{b,c\}) \\ &\cup \nmid_{\{b\}}(\mathcal{D}|\{c,a\})\end{aligned}$$

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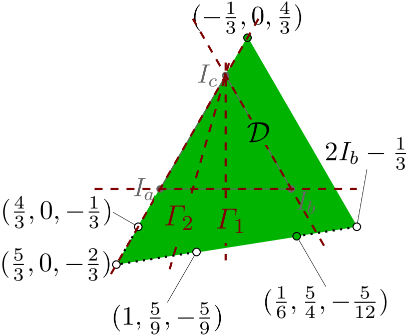

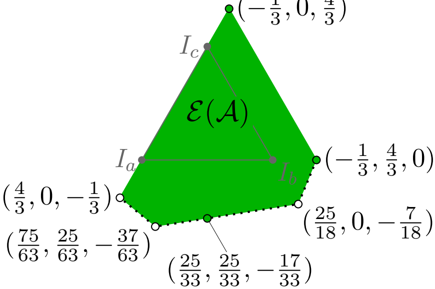

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Marginal extension

Separately specified conditional sets of desirable gambles have disjunct possibility spaces

Separately coherent conditional sets of desirable gambles are separately specified and individually coherent

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Marginal Extension Theorem

Given a partition $\mathcal{B}$ of $\mathcal{X}$, a coherent $\mathcal{B}$-marginal $\mathcal{D}_{\mathcal{B}} \subset \mathcal{L}(\mathcal{B})$, and separately coherent conditional sets of desirable gambles $\mathcal{D}|B \subset \mathcal{L}(B)$, $B \in \mathcal{B}$, then their combination $\mathcal{D} := \mathcal{E}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{X})$, with $\mathcal{A} := \mathit{\Gamma}_{\mathcal{B}}(\mathcal{D}_{\mathcal{B}}) \cup \bigcup_{B \in \mathcal{B}} \dagger_{B^c}(\mathcal{D}|B)$, is coherent as well.

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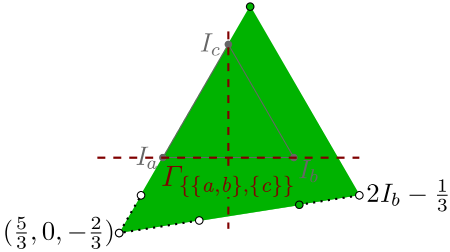

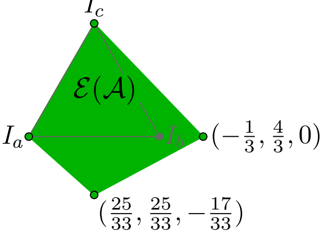

Exercises

1. Explicitly show that the transformation Γ_γ associated to the surjective map $\gamma : \{0,1\}^2 \to \{0,1,2\} : x \mapsto x_1 + x_2$ preserves coherence.
   1.1 What slice of $\mathcal{L}(\{0,1\}^2)$ does Γ_γ generate?
   1.2 What is the partition associated to γ?
2. Show that the transformation $\Gamma : \mathcal{L}(\{0,1,2\}) \to \mathcal{L}([0,1])$ that maps a gamble g to the parabola $g(0)(1-\theta)^2 + g(1)\theta(1-\theta) + 2g(2)\theta^2$ in θ does not preserve coherence, by considering $1 - 4\theta + 4\theta^2$.
   2.1 Describe the linear subspace of $\mathcal{L}([0,1])$ generated by Γ.
   2.2 Define a vector ordering on this subspace that makes Γ preserve coherence.
3. Take $\mathcal{E}(\mathcal{A}_7)$ from Exercise 2.2 of the previous series.
   3.1 Calculate its conditionals for all nonempty events of $\{a,b,c\}$, give the extreme-ray representation in all three formats.
   3.2 Calculate its marginals for all partitions of $\{a,b,c\}$.
   3.3 Calculate the marginal extensions of the appropriate derived conditionals and marginals for all partitions of $\{a,b,c\}$.
4. Prove the Marginal Extension Theorem.

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Partial preference orders

- Strict preference
- Nonstrict preference
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Partial strict preference order

Strict preference $f \succ g$ if we are eager to exchange g for f

Partial ... order The order does not have to be complete, $f \not\succ g \wedge g \not\succ f$ is possible

Strict desirability is strict preference over status quo, the zero gamble 0:

$$f \succ g \Leftrightarrow f - g \succ 0 \Leftrightarrow f - g \in \mathcal{D}$$

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Rationality criteria for strict preference relations $\succ$ on $\mathcal{L}(\mathcal{X}) \times \mathcal{L}(\mathcal{X})$:

Irreflexivity:	$f \not\succ f$
Transitivity:	$f \succ g \wedge g \succ h \Rightarrow f \succ h$
Mix-indep.:	$0 < \mu \leq 1 \Rightarrow \big(f \succ g \Leftrightarrow \mu f + (1-\mu)h \succ \mu g + (1-\mu)h\big)$
Monotonicity:	$f > g \Rightarrow f \succ g$

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Strengthening coherence criteria for sets of desirable gambles $\mathcal{D}$:

Avoiding nonpositivity: $f \leq 0 \Rightarrow f \notin \mathcal{D}$

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Strengthening coherence criteria for sets of desirable gambles $\mathcal{D}$:

Accepting nonnegativity: $f \geq 0 \Rightarrow f \in \mathcal{D}$

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Strengthening coherence criteria for sets of desirable gambles $\mathcal{D}$:

Accepting nonnegativity: $\mathcal{L}_0^+(\mathcal{X}) \subseteq \mathcal{D}$

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- Advantages of nonstrict preference:

  Indifference is the equivalence relation defined by symmetric nonstrict preference:

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  Incomparability is the irreflexive relation defined by symmetric nonstrict nonpreference:

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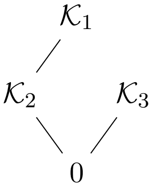

Example:

- $\equiv$-equivalence classes $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$
- intransitivity of $\bowtie$:
  $\mathcal{K}_1 \bowtie \mathcal{K}_3$ and $\mathcal{K}_3 \bowtie \mathcal{K}_2$, but $\mathcal{K}_1 \trianglerighteq \mathcal{K}_2$

Nonstrict preferences implied by strict ones

Motivation Indifference and incomparability are useful concepts

Associate a nonstrict preference relation $\succcurlyeq$ to a strict one $\succ$;
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Better proposal 'Making a sweet deal by sweetening an OK deal':

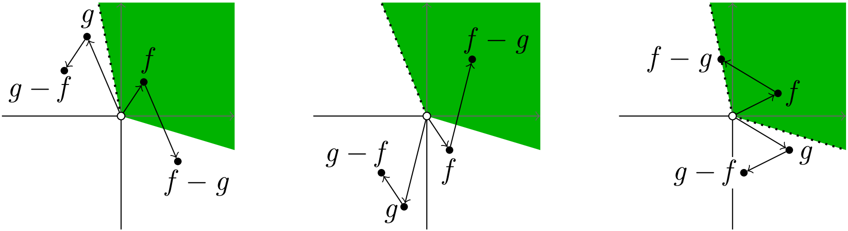
$$f \succcurlyeq g \Leftrightarrow f - g \succcurlyeq 0 \Leftrightarrow (f - g) + \mathcal{D}_{\succ} \subseteq \mathcal{D}_{\succ}$$

Immediate consequence:

$$f \succ g \Rightarrow g \not\succcurlyeq f$$

Incomparability $\asymp$ and indifference $\approx$

Strict and the associated nonstrict preferences: examples



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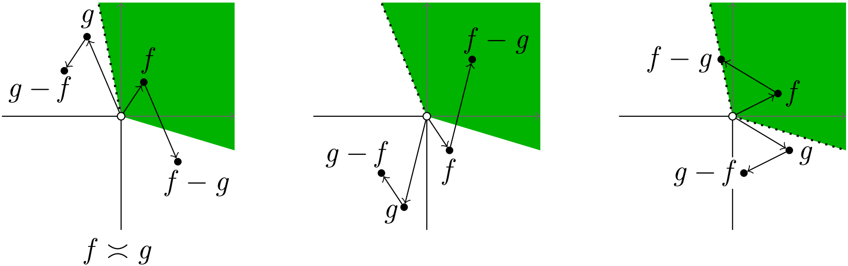

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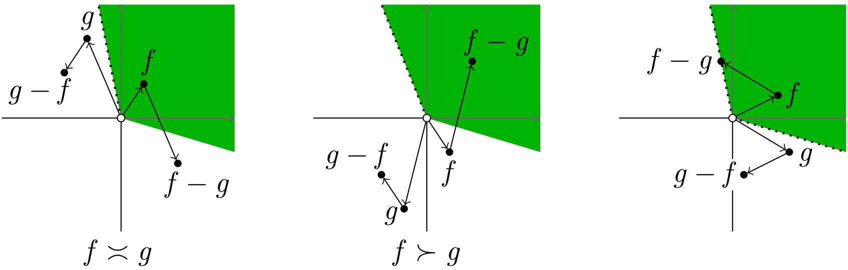

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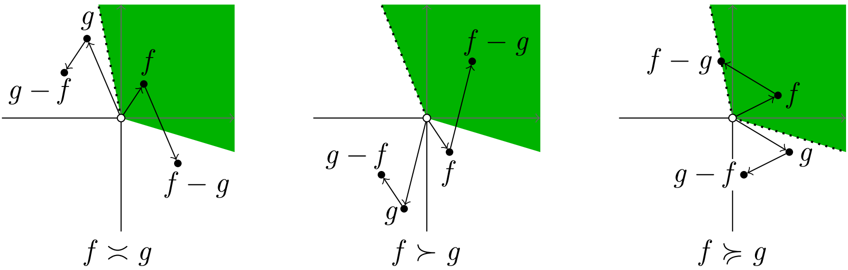

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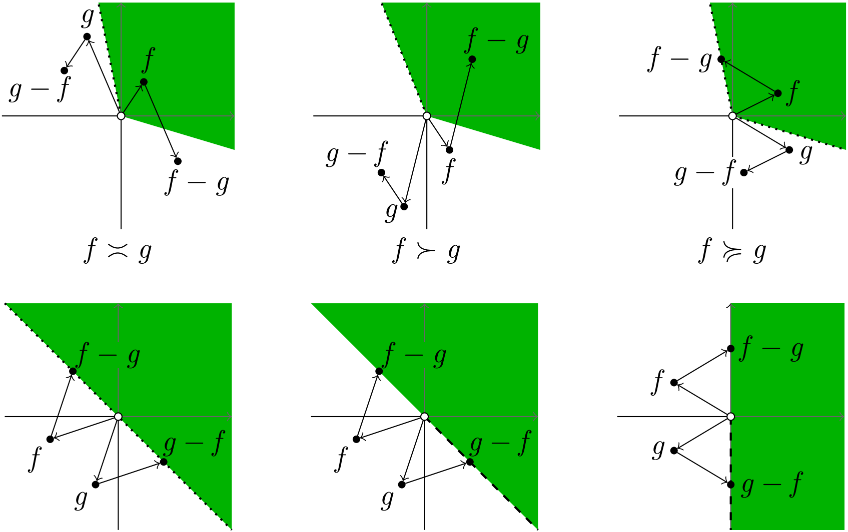

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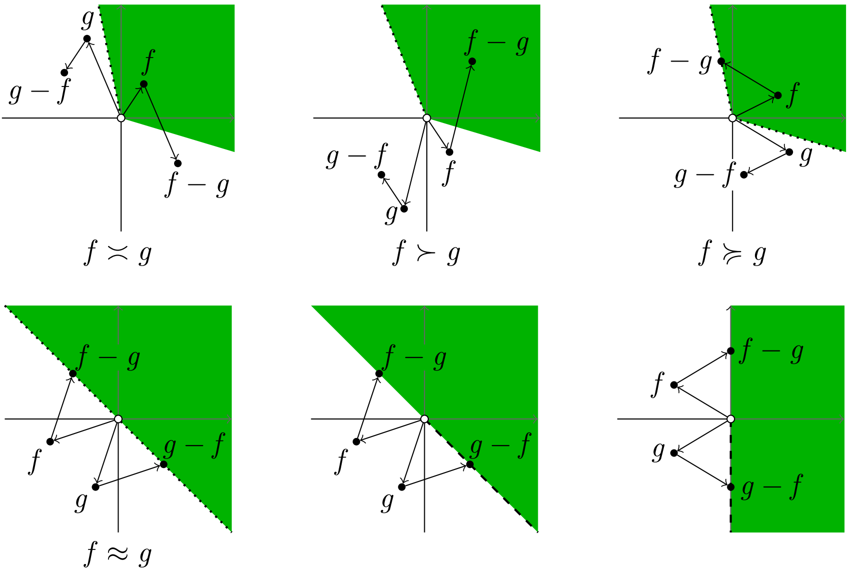

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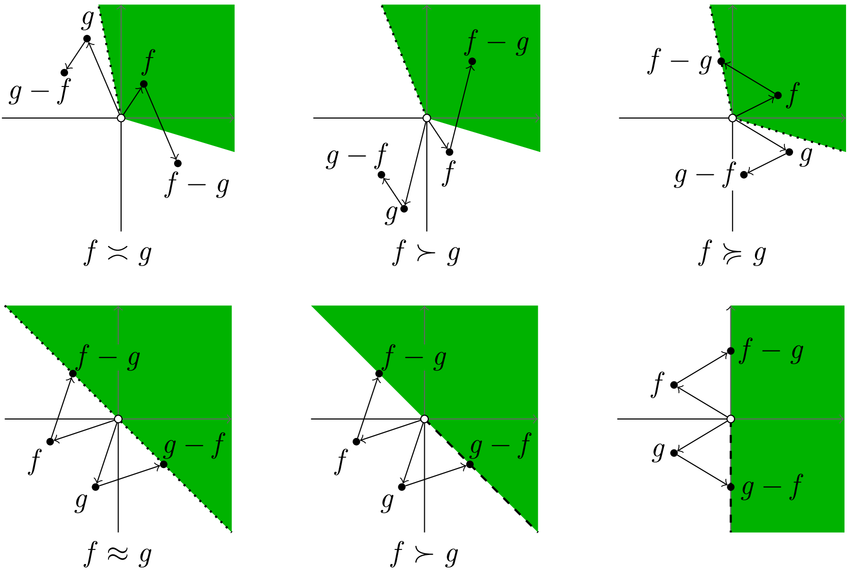

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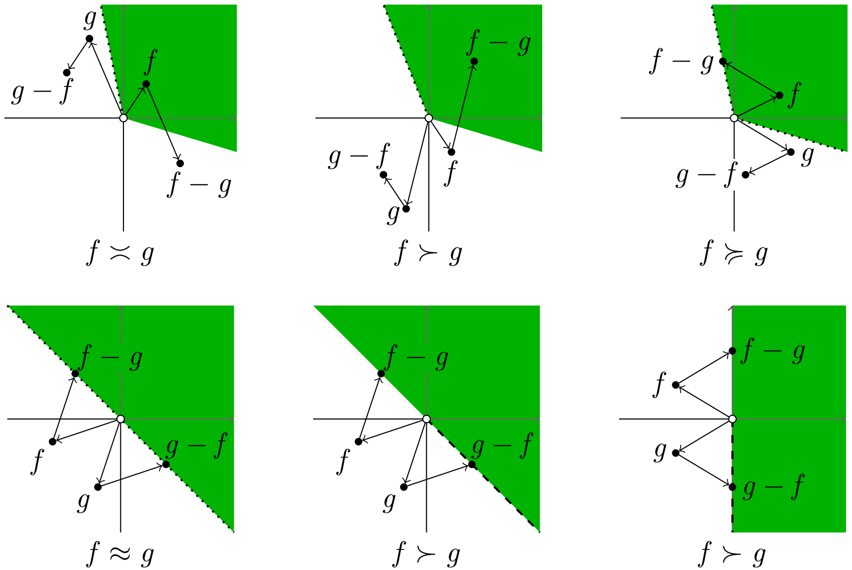

Strict preferences implied by nonstrict ones

Motivation Strict preferences are useful for decision making

Associate a strict preference relation $\rhd$ to a nonstrict one $\unrhd$;
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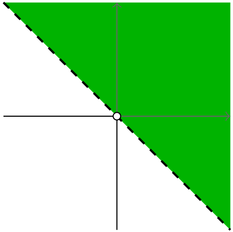

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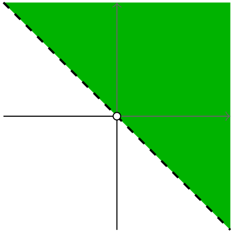

Other options? Not pursued: no proliferation of interpretations

We continue with strict desirability as the primitive notion

Exercises

1. Possibility space $\{a, b\}$.
   1.1 Which of $(-4, 3)$, $(-3, 4)$, and $(3, -3)$ belong to $\mathcal{D}_{\succ}$, $\mathcal{D}_{\succcurlyeq}$, both, or neither, when $(5, -2) \approx (-2, 5)$.
   1.2 Which, or both, or neither of $\{(-1, 1)\}$ and $\{(2, -3)\}$ is compatible as an assessment with $(5, -3) \asymp (4, -1)$.
2. Prove the equivalence of the rationality criteria for strict preference and strict desirability.
3. Prove that $\succcurlyeq$ satisfies the rationality criteria of nonstrict preference (assume they are equivalent to those for nonstrict desirability).

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- Maximally committal coherent extensions
- Maximality & transformations

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Maximal coherent sets of (strictly) desirable gambles ...

- are the maximal elements of $\mathbb{D}(\mathcal{X})$ ordered by inclusion
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Characterization of Maximal Sets of Desirable Gambles

The set $\mathcal{D}$ in $\mathbb{D}(\mathcal{X})$ is maximal if and only if $f \in \mathcal{D} \Leftrightarrow -f \notin \mathcal{D}$ for all nonzero gambles f on $\mathcal{X}$.

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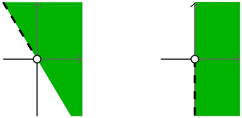
Characterization of Maximal Sets of Desirable Gambles

The set $\mathcal{D}$ in $\mathbb{D}(\mathcal{X})$ is maximal if and only if $f \in \mathcal{D} \Leftrightarrow -f \notin \mathcal{D}$ for all nonzero gambles f on $\mathcal{X}$.

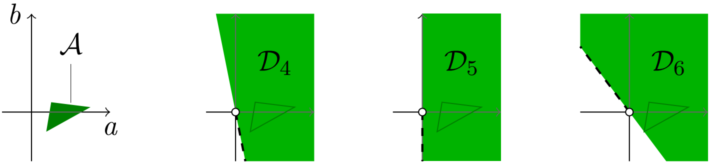
- are halfspaces that are neither open nor closed



- belong to the set $\hat{\mathbb{D}}(\mathcal{X})$

Maximally committal coherent extensions

Maximal coherent extension of an assessment $\mathcal{A} \subseteq \mathcal{L}(\mathcal{X})$ Any encompassing maximally committal coherent set of desirable gambles



Set of maximal coherent extensions $\hat{\mathbb{D}}_{\mathcal{A}} \coloneqq \{\mathcal{D} \in \hat{\mathbb{D}}(\mathcal{X}) \colon \mathcal{A} \subseteq \mathcal{D}\}$

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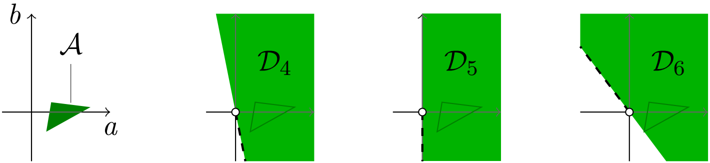

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Maximal Sets and Nonpositivity Avoidance Theorem

An assessment $\mathcal{A} \subseteq \mathcal{L}(\mathcal{X})$ avoids nonpositivity if and only if $\hat{\mathbb{D}}_{\mathcal{A}} \neq \emptyset$.

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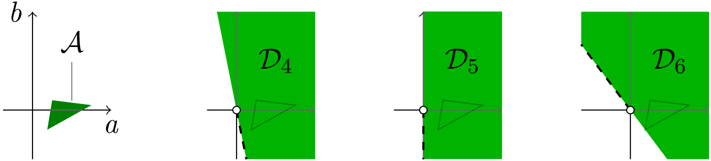

Set of maximal coherent extensions $\hat{\mathbb{D}}_{\mathcal{A}} := \{\mathcal{D} \in \hat{\mathbb{D}}(\mathcal{X}) \colon \mathcal{A} \subseteq \mathcal{D}\}$

Maximal Sets and Nonpositivity Avoidance Theorem

An assessment $\mathcal{A} \subseteq \mathcal{L}(\mathcal{X})$ avoids nonpositivity if and only if $\hat{\mathbb{D}}_{\mathcal{A}} \neq \emptyset$.

Maximal Sets and Natural Extension Corollary

The least committal extension of an assessment $\mathcal{A} \subseteq \mathcal{L}(\mathcal{X})$ that avoids nonpositivity, i.e., its natural extension $\mathcal{E}(\mathcal{A})$, is the intersection $\bigcap \hat{\mathbb{D}}_{\mathcal{A}}$ of the encompassing maximal sets of desirable gambles.

Maximality & transformations

Maximality Preserving Transformations Proposition

A coherence preserving transformation preserves maximality.

Exercises

1. Possibility space $\{a, b, c\}$; let $f := (-1, 1, 1)$ be an extreme ray of a maximal set of desirable gambles.
   1.1 Draw the intersection with the sum-one plane of the ones for which respectively $f + I_b - I_a$ and $f + I_c - I_a$ are nonstrictly desirable.
   1.2 Also draw their intersection with the sum-minus one plane.
2. Prove the Characterization of Maximal Sets of Desirable Gambles
3. Prove the Maximal Sets and Natural Extension Corollary
4. Prove the Maximality Preserving Transformations Proposition

Outline

Reasoning about and with sets of desirable gambles

Derived coherent sets of desirable gambles

Combining sets of desirable gambles

Partial preference orders

Maximally committal sets of strictly desirable gambles

Relationships with other, nonequivalent models

- Linear previsions
- Credal sets
- To lower & upper previsions
- Simplified variants of desirability
- From lower previsions
- Conditional lower previsions

Linear previsions

Linear previsions ...

- are positive linear normed expectation operators
- provide fair prices for gambles in $\mathcal{L}(\mathcal{X})$
- are equivalent to (finitely additive) probability measures and, on finite $\mathcal{X}$, to probability mass functions

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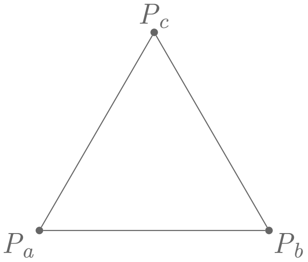

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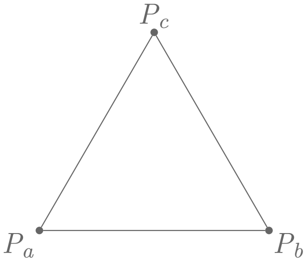

- provide probabilities for events, as fair prices for their indicators

From linear previsions to sets of desirable gambles

Given a linear prevision $P \in \mathbb{P}(\mathcal{X})$, gambles with a strictly positive fair price are strictly desirable:

$$\mathcal{D}_P := \mathcal{E}(\mathcal{A}_P), \quad \text{with} \quad \mathcal{A}_P := \{f \in \mathcal{L}(\mathcal{X}) \colon P(f) > 0\}$$

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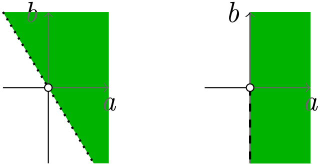
Observations:

- $\{f \in \mathcal{L}(\mathcal{X}) \colon P(f) = 0\}$ is a linear subspace of $\mathcal{L}(\mathcal{X})$
- So $\mathcal{A}_P$ is an open halfspace
- Except in a few borderline cases, so is $\mathcal{D}_P$



- Except in two nontrivial cases, $\mathcal{D}_P$ is nonmaximal, so $\hat{\mathbb{D}}_P \subseteq \mathbb{D}_P$ are nontrivial

From credal sets to sets of desirable gambles

A credal set is a set of linear previsions

Given a credal set $\mathcal{M} \subseteq \mathbb{P}(\mathcal{X})$, gambles with a strictly positive fair price for every linear prevision in the credal set are strictly desirable:

$$\mathcal{D}_{\mathcal{M}} \coloneqq \mathcal{E}(\mathcal{A}_{\mathcal{M}}), \quad \text{with} \quad \mathcal{A}_{\mathcal{M}} \coloneqq \{f \in \mathcal{L}(\mathcal{X}) \colon (\forall P \in \mathcal{M} : P(f) > 0)\} = \bigcap_{P \in \mathcal{M}} \mathcal{A}_P$$

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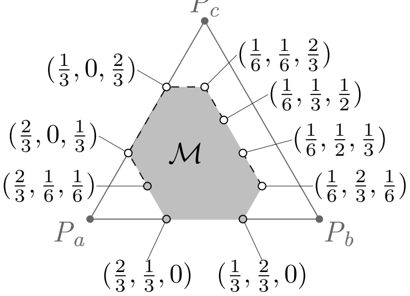
Observations:

- Each prevision gives rise to a linear constraint in gamble space
- Constraints from linear previsions strictly in the convex hull of $\mathcal{M}$ are redundant
- So the border structure of $\mathcal{M}$ is uniquely important

From credal sets to sets of desirable gambles: example



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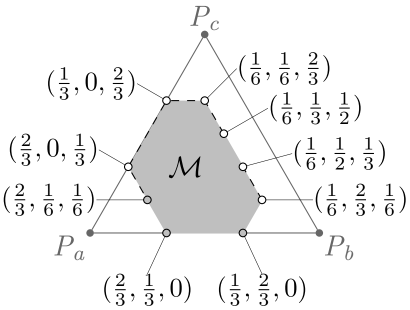
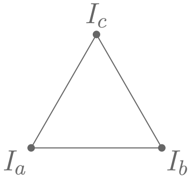
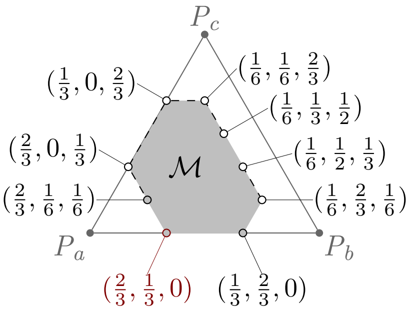
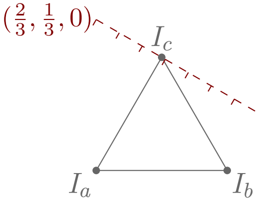




From credal sets to sets of desirable gambles: example

$$\frac{2}{3}f(a) + \frac{1}{3}f(b) > 0$$





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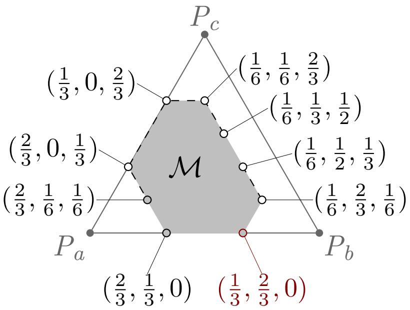

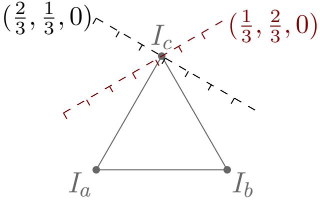

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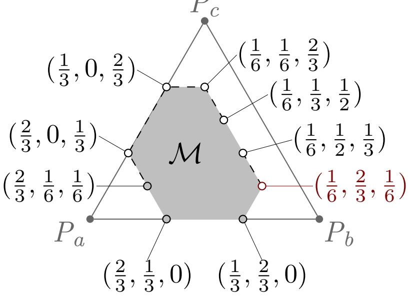

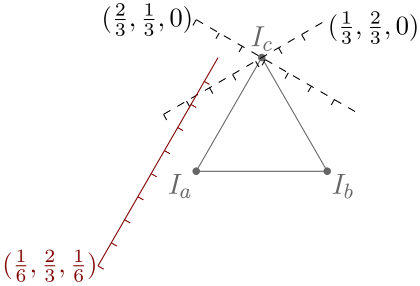

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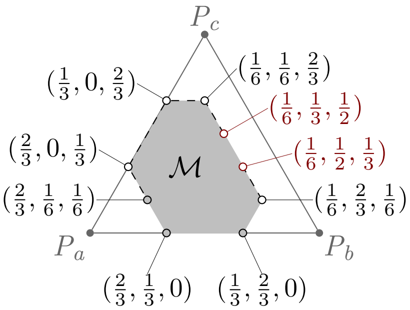

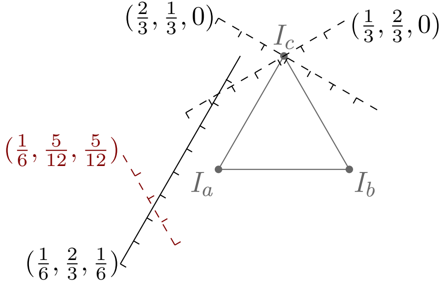

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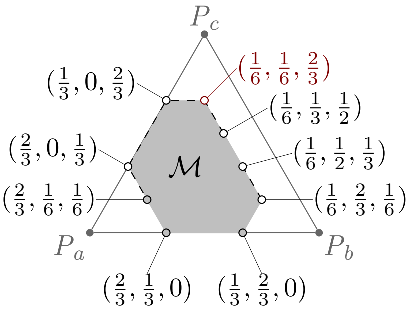

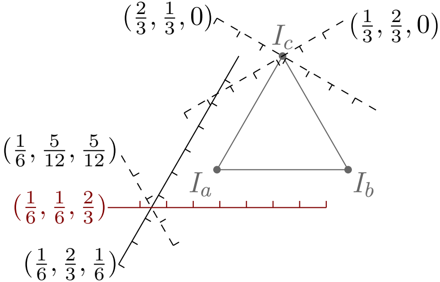

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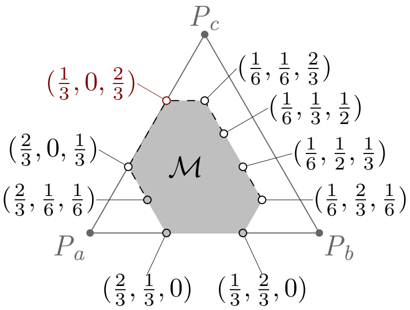

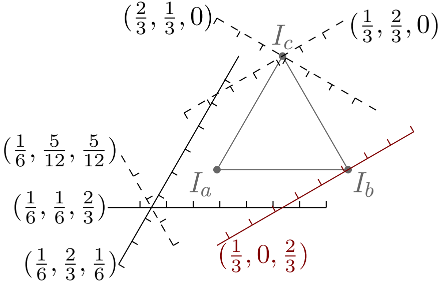

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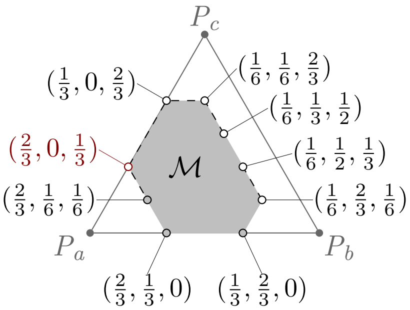

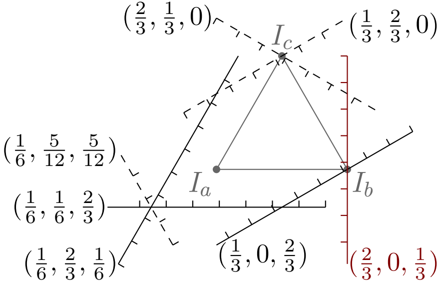

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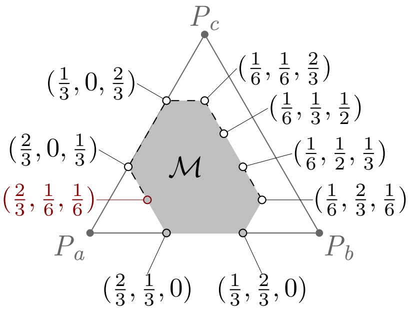

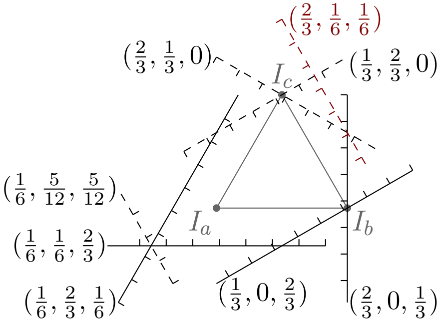

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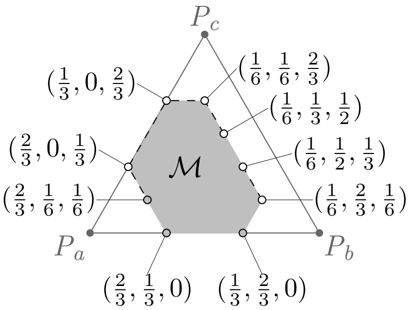

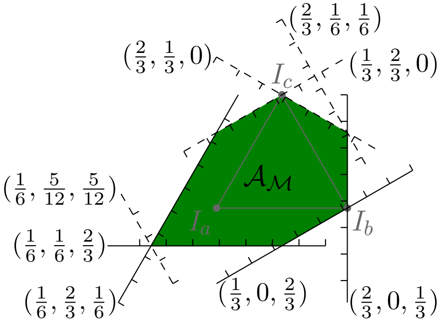

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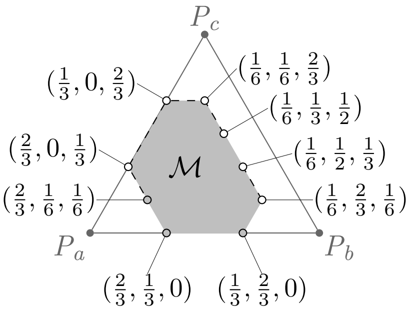
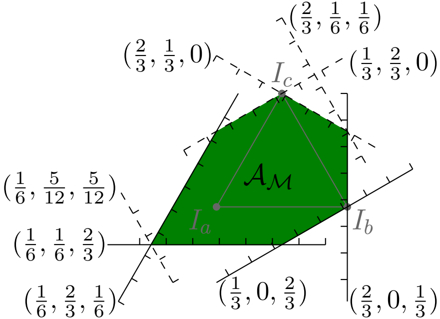
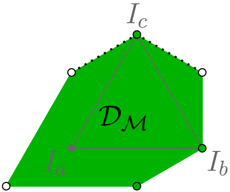






From desirable gambles to credal sets

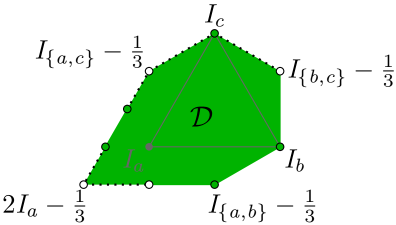
Given a coherent set of strictly desirable gambles $\mathcal{D} \subset \mathcal{L}(\mathcal{X})$, we use its set of (maximally committal) coherent extensions to derive the associated credal set:

$$\begin{aligned} \mathcal{M}_{\mathcal{D}} &:= \{P \in \mathbb{P}(\mathcal{X}) \colon \mathbb{D}_P \cap \mathbb{D}_{\mathcal{D}} \neq \emptyset\} \\ &= \{P \in \mathbb{P}(\mathcal{X}) \colon \hat{\mathbb{D}}_P \cap \hat{\mathbb{D}}_{\mathcal{D}} \neq \emptyset\} \end{aligned}$$

Credal Set Conjecture

The credal set $\mathcal{M}_{\mathcal{D}} \subseteq \mathbb{P}(\mathcal{X})$ associated to a coherent set of desirable gambles $\mathcal{D} \subset \mathcal{L}(\mathcal{X})$ is closed and convex.

From desirable gambles to credal sets: example



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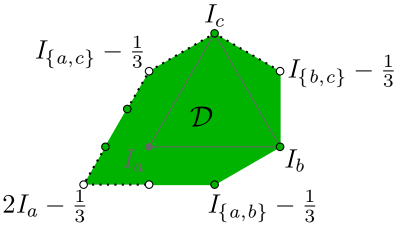

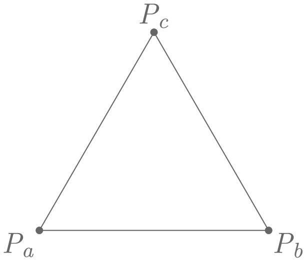

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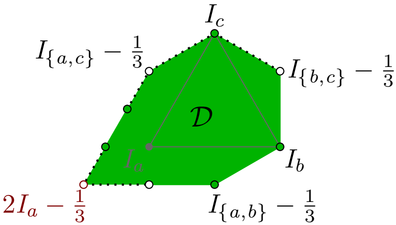

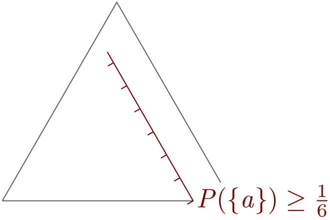

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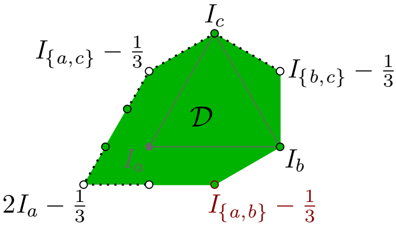

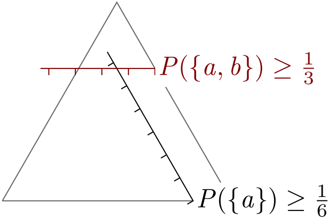

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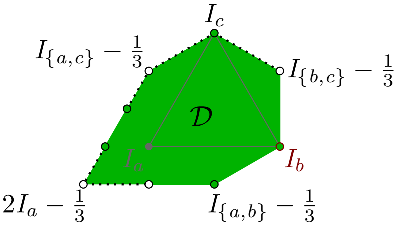

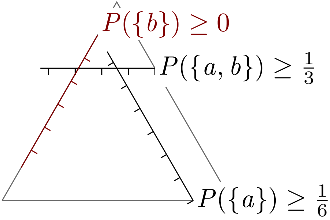

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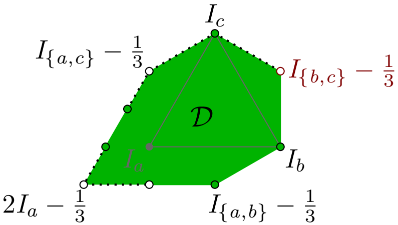

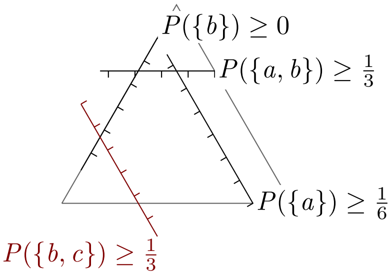

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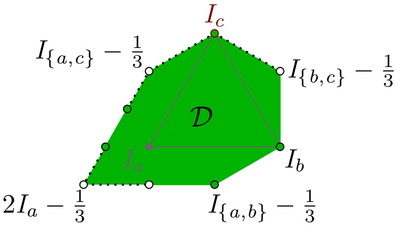

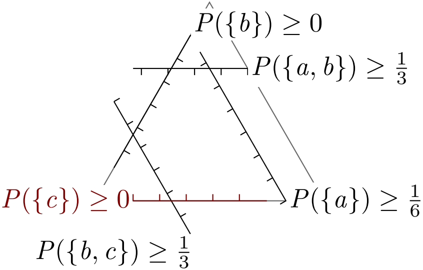

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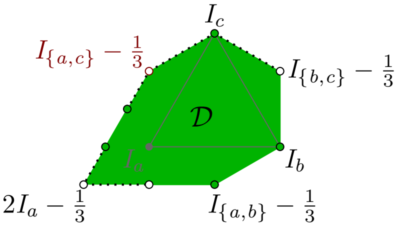

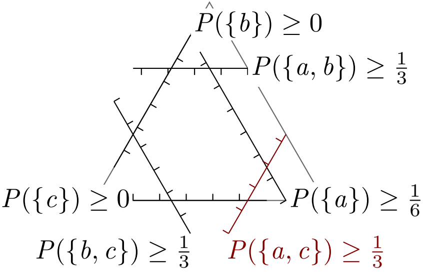

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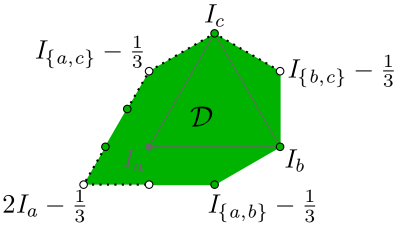

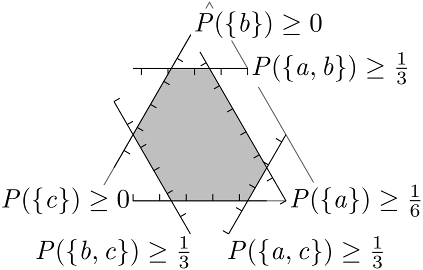

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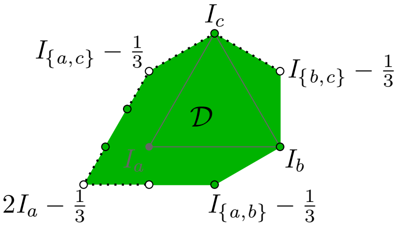

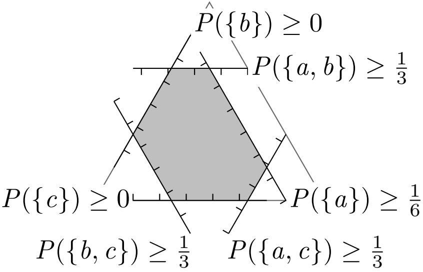

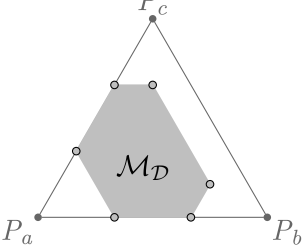

Lower & upper previsions

Lower previsions . . .

- are positive superlinear normed expectation operators
- provide supremum acceptable buying prices for gambles in $\mathcal{L}(\mathcal{X})$
- provide lower probabilities for events

Upper previsions . . .

- are positive sublinear normed expectation operators
- provide infimum acceptable selling prices for gambles in $\mathcal{L}(\mathcal{X})$
- provide upper probabilities for events

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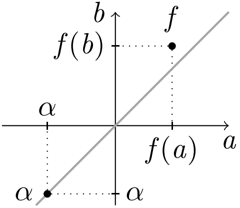
Prices can be seen as constant gambles, which are trivially linearly ordered



From sets of desirable gambles to lower & upper previsions

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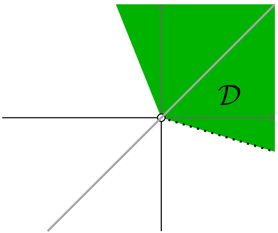

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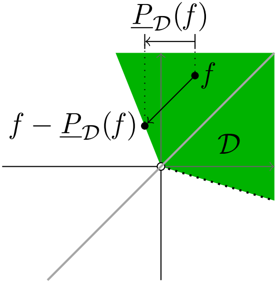

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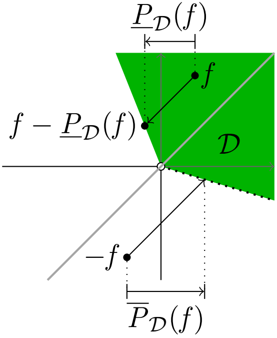

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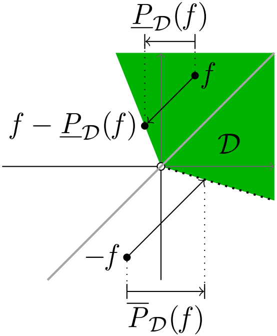

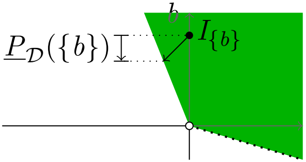

Conjugacy: $\overline{P}_{\mathcal{D}}(f) = -\underline{P}_{\mathcal{D}}(-f)$ and $\overline{P}_{\mathcal{D}}(A) = 1 - \underline{P}_{\mathcal{D}}(A^c)$

Simplified variants of desirability

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Simplified variants of desirability: relationships & example

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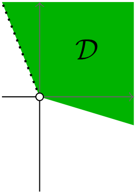

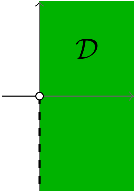

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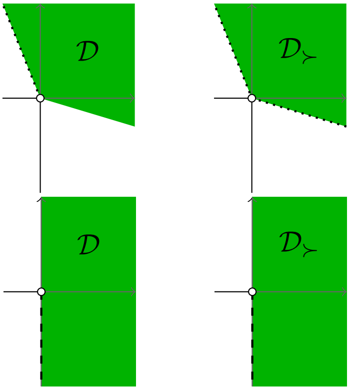

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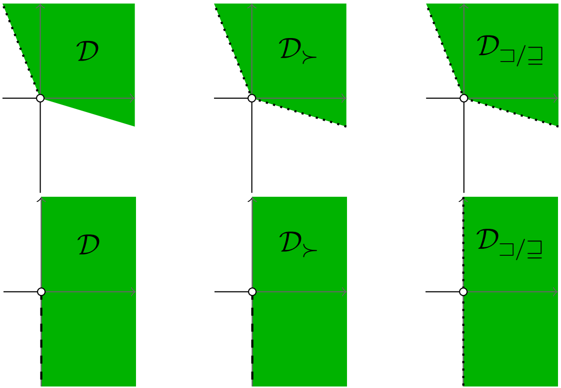

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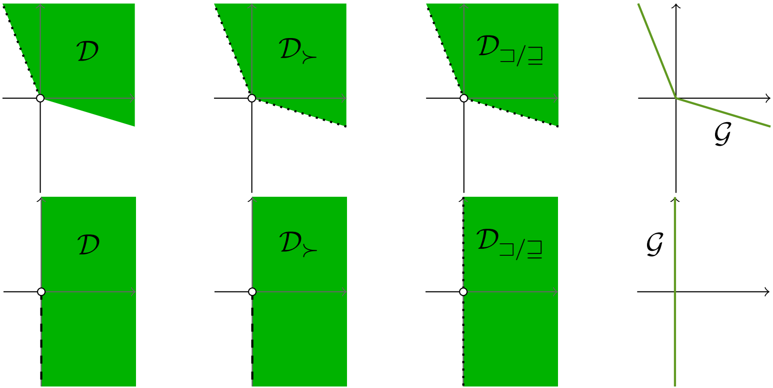

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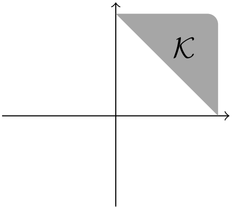

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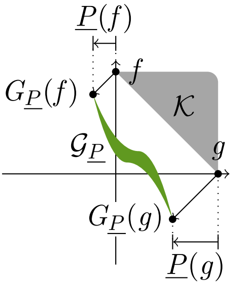

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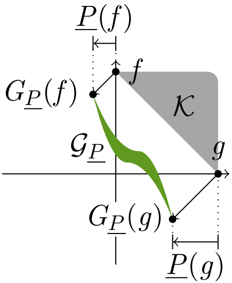

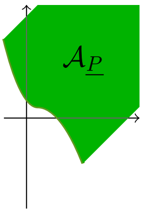

Translating desirability concepts to lower previsions

Avoiding sure loss for a lower prevision $\underline{P}$ on $\mathcal{K} \subseteq \mathcal{L}(\mathcal{X})$ corresponds to $\mathcal{A}_{\underline{P}}$ avoiding sure (or partial) loss:

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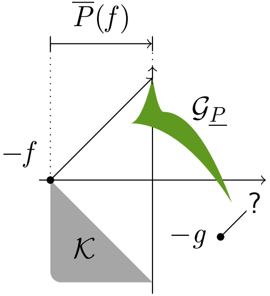

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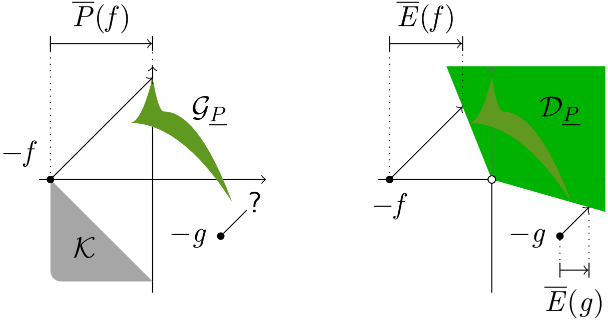

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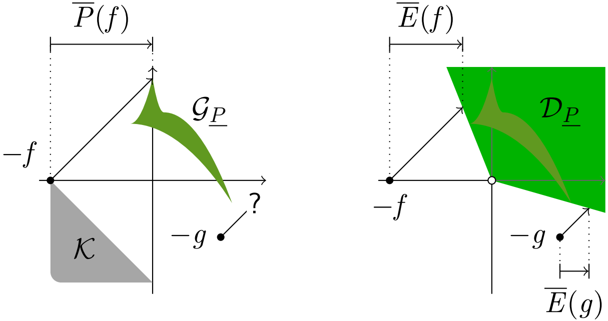

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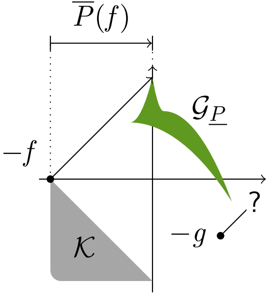

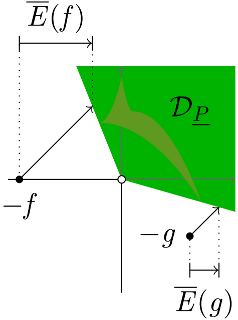

Coherence for lower previsions $\underline{P}$ on $\mathcal{K} \subseteq \mathcal{L}(\mathcal{X})$ corresponds to coherence of $\mathcal{D}_{\underline{P}}$:

$$\forall f \in \mathcal{G}_{\underline{P}} : \forall g \in \operatorname{posi}(\mathcal{G}_{\underline{P}}) : \sup(g - f) \geq 0$$

Natural versus regular extension

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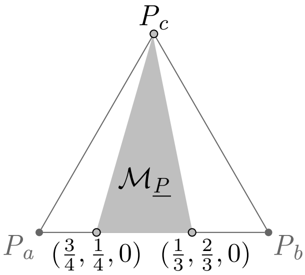

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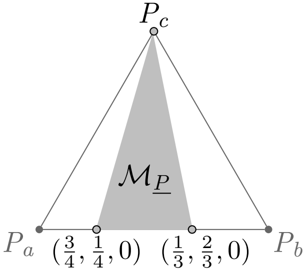

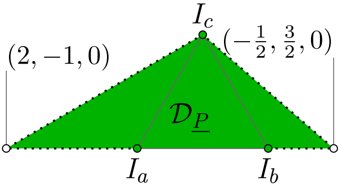

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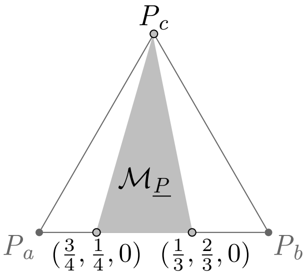

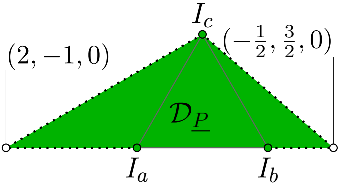

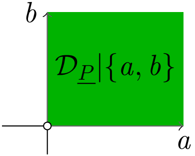

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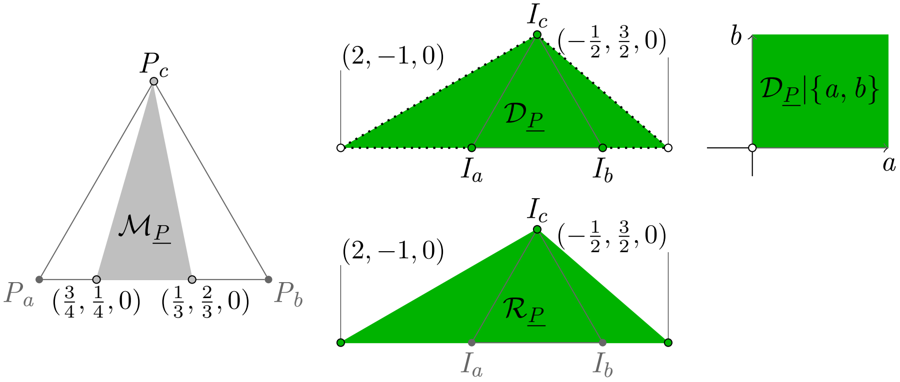

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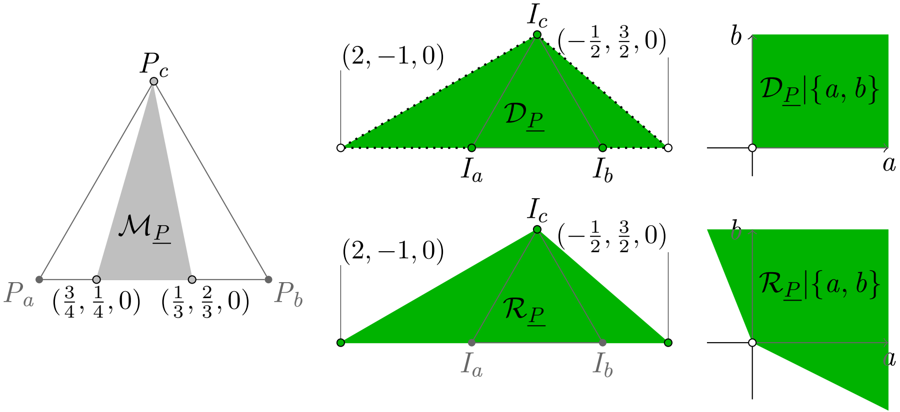

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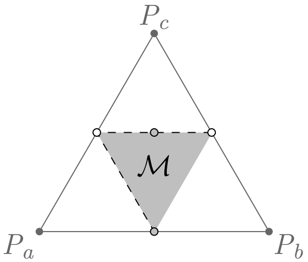
Exercises I

1. Possibility space $\{a, b, c\}$; draw the intersection of $\mathcal{D}_{P_i}$ with the sum-one and sum-minus one planes for the linear previsions defined by

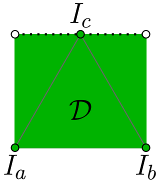
$$P_1(f) = \tfrac{1}{2}f(a) + \tfrac{1}{4}f(b) + \tfrac{1}{4}f(c) \quad \text{and} \quad P_2(f) = \tfrac{1}{3}f(a) + \tfrac{2}{3}f(b)$$

2. Calculate the set of desirable gambles $\mathcal{D}_{\mathcal{M}}$ corresponding to the given credal set $\mathcal{M}$:



Exercises II

3. Calculate the credal set $\mathcal{M}_{\mathcal{D}}$ corresponding to the given set of desirable gambles $\mathcal{D}$:



4. Give the corresponding simplified variants for all the sets of desirable gambles appearing up until now in this exercise series.
5. Possibility space $\{a, b, c\}$; a lower prevision $\underline{P}$ is specified as follows: the lower probability of $\{c\}$ and $\{b, c\}$ are, respectively, $\frac{1}{6}$ and $\frac{1}{4}$; the supremum upper buying price for $(-3, 3, -2)$ is -2.
   - 5.1 Calculate $\mathcal{D}_{\underline{P}}$ and use it to check ...
   - 5.2 whether $\underline{P}$ avoids sure loss,
   - 5.3 whether $\underline{P}$ is coherent,
   - 5.4 calculate the natural extension of $\underline{P}$ to $I_{\{a,b\}}$, $I_{\{b,c\}}$, and $I_{\{c,a\}}$.

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Extra material: Conglomerability

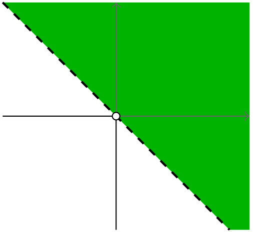
Some authors require *full conglomerability* as a coherence criterion for sets of desirable gambles $\mathcal{D} \subseteq \mathcal{L}(\mathcal{X})$, which is conglomerability relative to all partitions $\mathcal{B}$ of $\mathcal{X}$:

$$\mathcal{B}\text{-Conglomerability:} \quad (\forall B \in \mathcal{B} : fI_B \in \mathcal{D}) \Rightarrow f \in \mathcal{D}$$

This is of importance for deriving conditional sets of desirable gambles separately specified on infinite partitions

Extra material: Lexicographic models

Can we make sense of mostly open cones of nonstrictly desirable gambles?



Extra material: Lexicographic models

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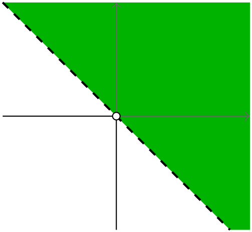

We can look at it as a partial view of a more complex uncertainty model:

Infinitesimal precision is used when defining payoffs

Lexicographic utility can be used for finite possibility spaces
(2-tier for this example)

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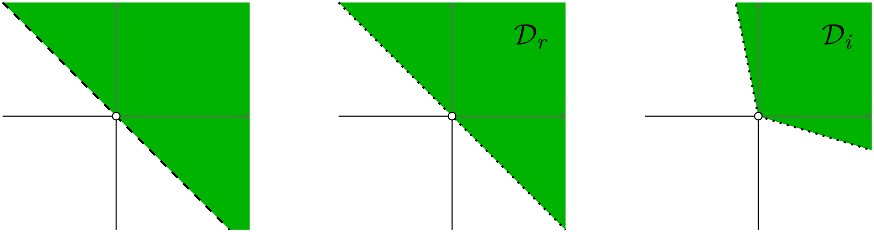

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- lexicographic gamble $h := h_r + \epsilon h_i$, with ϵ an infinitesimal quantity and h_r and h_i real-valued
- set of desirable lexicographic gambles $\mathcal{D} := \mathcal{D}_r + \epsilon \mathcal{D}_i$

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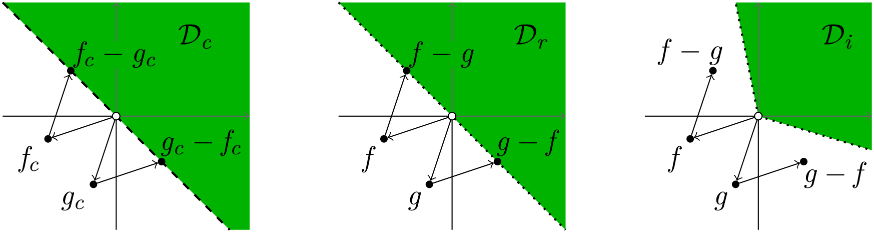

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Infinitesimal precision is used when defining payoffs

Lexicographic utility can be used for finite possibility spaces (2-tier for this example)

- lexicographic gamble $h \coloneqq h_r + \epsilon h_i$, with ϵ an infinitesimal quantity and h_r and h_i real-valued
- set of desirable lexicographic gambles $\mathcal{D} \coloneqq \mathcal{D}_r + \epsilon \mathcal{D}_i$
- original shows lexicographic gambles that are constant over the tiers: $f_c \coloneqq f + \epsilon f$, with f real-valued

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Lexicographic probability, conditional probability, and nonstandard probability.
Games and Economic Behavior, 68(1):155–179, 2010.

Full section outline

Reasoning about and with sets of desirable gambles

Derived coherent sets of desirable gambles

Combining sets of desirable gambles

Partial preference orders

Maximally committal sets of strictly desirable gambles

Relationships with other, nonequivalent models