A constrained optimization problem under uncertainty

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Toy problem: two-component massless rod

\[
Y_1 \quad Y_2
\]

\[
l_1 = (1 - x)L \quad l_2 = xL
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Goal: Maximize \(x\) under the constraint that \(d_2 < D\).
Toy problem: two-component massless rod, tensile load

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Two-component massless rod, tensile load: FE analysis

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FE analysis 3 nodes, boundary conditions

\[
\begin{bmatrix}
  c_1 + c_2 & -c_2 \\
  -c_2 & c_2 
\end{bmatrix}
\begin{bmatrix}
  d_1 \\
  d_2 
\end{bmatrix} = \begin{bmatrix}
  0 \\
  F 
\end{bmatrix}, \quad c_i = \frac{Y_i a}{l_i}.
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Solution solving the system (analytically) gives

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d_1 = \frac{FL}{a} \frac{1-x}{Y_1}, \quad d_2 = d_1 + \frac{FL}{a} \frac{x}{Y_2}.
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Goal Maximize \(x\) under the constraint that \(\frac{1-x}{Y_1} + \frac{x}{Y_2} < \frac{D a}{F L}\).

Goal Maximize \(x\) under the constraint that \(d_2 < D\).
Two-component rod, tensile load: design optimization

Precisely known elastic moduli $Y_1$ and $Y_2$ This problem is ▶ a classical constrained optimization problem; ▶ considered 'solved'.

Uncertainty about elastic moduli $Y_1$ and $Y_2$ This problem is ▶ a constrained optimization problem under uncertainty; ▶ not well-posed as such.

Approach: ▶ reformulate as a well-posed decision problem; ▶ solve the decision problem, i.e., derive a classical constrained optimization problem.

\[
\begin{align*}
\text{l} = (1-x)L & \quad \text{l}_2 = xL \\
F & \quad d_1 \\
F & \quad d_2
\end{align*}
\]

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\frac{1-x}{Y_1} + \frac{x}{Y_2} < \frac{Da}{FL}.
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Overview

Toy problem

General problem formulation

Uncertainty models

Optimality criteria

Probabilistic and indeterminacy aspects of uncertainty

Objective

Results

Application: bridge design for vehicle-pillar collisions
A constrained optimization problem under uncertainty

Goal Maximize $f(x)$ under the constraint that $xRY$.

- $x$ optimization variable (values in $X$)
- $f$ objective function (from $X$ to $\mathbb{R}$)
- $Y$ random variable (realizations $y$ in $Y$)
- $R$ relation on $X \times Y$.
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Decision problem Find the optimal decisions $x$:
- associate a utility function with every decision $z$:

$$G_z(y) = f(z)I_{zR} + LI_{zR} = \begin{cases} f(z), & zRy, \\ L, & z\notR y, \end{cases}$$

with penalty value $L < \inf f$;
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with penalty value $L < \inf f$;

- choose an optimality criterion, e.g., maximinity, maximality.
Goal  Faced with uncertainty about $y$ in $\mathcal{Y}$, find optimal $x$ in $\mathcal{X}$ given an optimality criterion and utility functions $G_z$ on $\mathcal{Y}$ for all $z$ in $\mathcal{X}$. 

Uncertainty models

Formal model for the uncertainty about $y$ in $\mathcal{Y}$.

Lower and upper expectation With (almost) all typical uncertainty models correspond lower and upper expectation operators ($E$ and $E^*$), or (almost) equivalently, a set of linear expectation operators $\mathcal{M}$:

$$E_{\mathcal{M}}(G) := \inf_{E \in \mathcal{M}} E(G),$$

$$E^*_{\mathcal{M}}(G) := \sup_{E \in \mathcal{M}} E(G),$$

$\mathcal{M}_{E_{\mathcal{M}}} := \{E : E \geq E^*\}$.

Examples

▶ probabilities (measures, PMF, PDF, CDF);
▶ upper and/or lower of the above (inner/outer measures, Choquet capacities, p-boxes);
▶ intervals, vacuous expectations: $E_A(G) := \inf_{y \in A} G(y)$;
▶ possibility distributions, belief functions, ...
▶ convex mixtures of the lot (e.g., contamination models).
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Optimality criteria: maximizing expected utility generalized

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Optimality criteria: maximizing expected utility generalized

Maximinity  Worst-case reasoning; optimal $x$ maximize the lower (minimal) expected utility ($P(A) := E(I_A)$):

$$E(G_x) = \sup_{z \in X} E(G_z)$$
$$= \sup_{z \in X} E(f(z)I_{zR} + LI_{zR}) = L + \sup_{z \in X} (f(z) - L)P(zR).$$

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$$E(G_x) = \sup_{z \in \mathcal{X}} E(G_z) = \sup_{z \in \mathcal{X}} E(f(z)I_{z \in R} + LI_{z \in R}) = L + \sup_{z \in \mathcal{X}} (f(z) - L)P(z \in R).$$

Maximality  Optimal $x$ are undominated in pairwise comparisons with all other decisions:

$$0 \leq \inf_{z \in \mathcal{X}} E(G_x - G_z) = \inf_{z \in \mathcal{X}} E \left( (f(x) - f(z))I_{x \in R \cap z \in R} + (f(x) - L)I_{x \in R \cap z \notin R} + (L - f(z))I_{x \notin R \cap z \in R} \right).$$

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Others  Maximaxity, $E$-admissibility, interval dominance

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Probabilistic and indeterminacy aspects of uncertainty

Example $\mathcal{X} = \mathcal{Y} := \mathbb{R}$, $R := \leq$.

$$f(x)$$

$\sup f|_{R_y}$

$R_y$ $y$ $R_y$ $x$

$g_y(x) = G_x(y)$

$\sup g_y$

$L$

$y$ $x$
Probabilistic and indeterminacy aspects of uncertainty

Example $\mathcal{X} = \mathcal{Y} := \mathbb{R}$, $R := \leq$.

**Indeterminacy** Assume $y$ can be either $y_1$ or $y_2$, but nothing more is known.
Probabilistic and indeterminacy aspects of uncertainty

Example \( \mathcal{X} = \mathcal{Y} := \mathbb{R}, \ R := \leq \).

Probabilistic Assume that \( y_1 \) and \( y_2 \) are equally likely.
Objective, deliverables, and a disclaimer

Research objective  decision problem solutions for combinations of various uncertainty models and optimality criteria.

Deliverables  A solution toolbox for a specific, but quite general class of decision problems under uncertainty.
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Research objective  decision problem solutions for combinations of various uncertainty models and optimality criteria.

Deliverables  A solution toolbox for a specific, but quite general class of decision problems under uncertainty.

Disclaimer  No reduction in the computational complexity; one faces

- an optimization problem to find the uncertainty-independent constraints,
- the resulting classical constrained optimization problem.
Results: probabilities

Optimal decision when $Y$ is described by a probability $P$.

Maximizing expected utility

- General case:

$$\text{argsup}_{z \in X} (f(z) - L)P(zR).$$

- Example: $X = Y := \mathbb{R}$, $R := \leq$.

$$\text{argsup}_{z \in \mathbb{R}} (f(z) - L) \left( 1 - F(z) \right),$$

where $F_Y(x) := P(\mathbb{R}_{\leq x}) = 1 - P(x \leq)$

is a continuous CDF.
Results: vacuous models

Optimal decision when $Y$ is described by a vacuous lower expectation relative to $A \subseteq Y$.

Maximinity

- General case:

$$\text{argsup}_{z \in RA} f(z), \quad RA := \bigcap_{y \in A} Ry.$$

- Example: $\mathcal{X} = \mathcal{Y} := \mathbb{R}$, $R := \leq$, $A := [a, b]$.

$$\text{argsup}_{z \leq a} f(z).$$
Results: vacuous models

Optimal decision when $Y$ is described by a vacuous lower expectation relative to $A \subseteq \mathcal{Y}$.

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General case:

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$$\text{argsup}_{z \leq a} f(z).$$

Maximality

General case:

$$x \in RA \quad \text{such that} \quad f(x) = \sup_{z \in RA} f(z), \quad RA := \bigcup_{y \in A} Ry.$$  

Example: $\mathcal{X} = \mathcal{Y} := \mathbb{R}$, $R := \leq$, $A := [a, b]$.  

$$x \leq b \quad \text{such that} \quad f(x) \geq \sup_{z \leq a} f(z).$$
Optimal decision when $Y$ is described by a possibility distribution $\pi$ on $\mathcal{Y}$; $P(A) := 1 - \sup_{y \in \mathcal{Y} \setminus A} \pi(y)$.

Maximinity

- General case:

$$\arg\sup_{z \in \mathcal{X}} (f(z) - L) \left( 1 - \sup_{y \in z \leq R} \pi(y) \right).$$

- Example: $\mathcal{X} = \mathcal{Y} := \mathbb{R}$, $R := \leq$, continuous $\pi$ with minimal mode $c \in \mathbb{R}$.

$$\arg\sup_{z < c} (f(z) - L) \left( 1 - \pi(z) \right).$$
Bridge design: vehicle colliding into pillar

Vehicle parameters mass $m$, stiffness $k$, initial speed $v_0$, average deceleration $a$, and swerve angle $\alpha$.

Bridge parameters pillar design loads $F_\parallel$ (longitud.) and $F_\perp$ (perpendicular).

What is the optimal lateral distance $x$ between the vehicle and curb that ensures structural integrity?
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Structural integrity constraint

$$F_{\text{veh.}} \cos \alpha \leq F_\parallel, \quad F_{\text{veh.}} \sin \alpha \leq F_\perp, \quad F_{\text{veh.}} = \sqrt{mk(v_0^2 - 2ax / \sin \alpha)}.$$
Bridge design: optimization problem under uncertainty

**Goal** Choose an optimal $x$ under the constraint that $F_{\text{veh.}} \cos \alpha \leq F_\parallel$ and $F_{\text{veh.}} \sin \alpha \leq F_\perp$. 
Bridge design: optimization problem under uncertainty

Objective function Based on dimensions-dependent building costs:

\[ f(x) := -45B((L_1 + 2d)^2 + 2L_2^2), \]

where \( B = 14, L_1 = 33, L_2 = 15 \) for a typical 3-span bridge.

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Penalty value

\( L \) was difficult to assess, so a number of values between \(-10^{6.3}\) and \(-10^{8.6}\) were tried.

Parameters

\( k = 300 \) [kN/m]; \( Y = (m, v_0, a, \alpha) \), independent product.

Goal

Choose an optimal \( x \) under the constraint that

\( F_{\text{veh. cos } \alpha} \leq F_{\perp} \) and \( F_{\text{veh. sin } \alpha} \leq F_{\perp} \).
Bridge design: uncertainty models for the parameters

Mass $m$ [t]
- Lorry: normal with mean 20, standard deviation 12, and realistic range $[12, 40]$.
- Car: vacuous in the interval $[.5, 1.6]$.

Initial velocity $v_0$ [km/h]
- Highway: 80, 10, $[50, 100]$.
- Urban: 40, 8, $[30, 70]$.

Average deceleration $a$ [m/s$^2$] Lognormal 4, 1.3, $[1, 5]$.

Swerve angle $\alpha$ [$^\circ$] Normal 30, 3, $[8, 45]$.
Bridge design: maximinity results for different vehicle types

Lorry – Highway \( F_\parallel = 1000, \ F_\perp = 500 \)

Lorry – Urban \( F_\parallel = 500, \ F_\perp = 250 \)

Lorry – Courtyard \( F_\parallel = 150, \ F_\perp = 75 \)

Car – Courtyard \( F_\parallel = 50, \ F_\perp = 25 \)

Car – Parking \( F_\parallel = 40, \ F_\perp = 25 \)
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Bridge design: maximinity results for different vehicle types

Lorry – Highway  \( F_\parallel = 1000, F_\perp = 500, x = 42 \) for \( L = -10^8 \).

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Car – Parking \( F_\parallel = 40, F_\perp = 25, x = 1.8 \) for \( L = -10^7 \).
Bridge design: maximality results for different vehicle types

Car – Courtyard  \( F_\parallel = 50, \ F_\perp = 25, \ x \in [2.4, 4.0] \) for \( L = -12 \cdot 10^6 \).

Car – Parking  \( F_\parallel = 40, \ F_\perp = 25, \ x \in [0.4, 1.7] \) for \( L = -7 \cdot 10^6 \).