A constrained optimization problem under uncertainty

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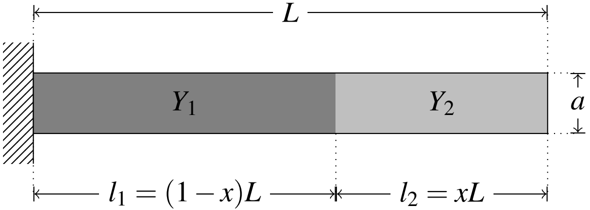
Toy problem: two-component massless rod



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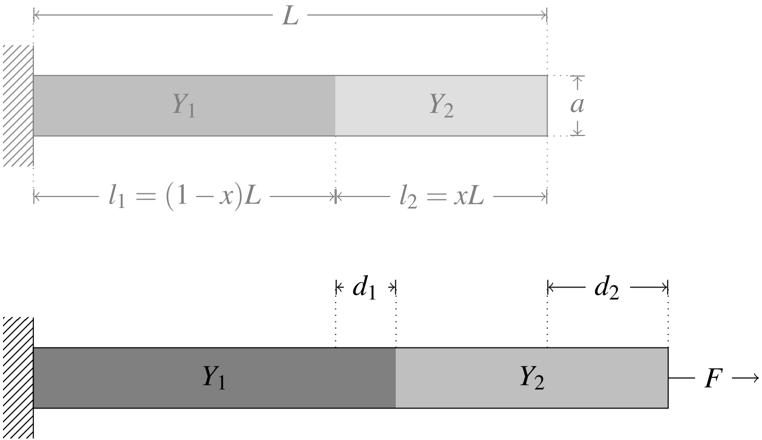

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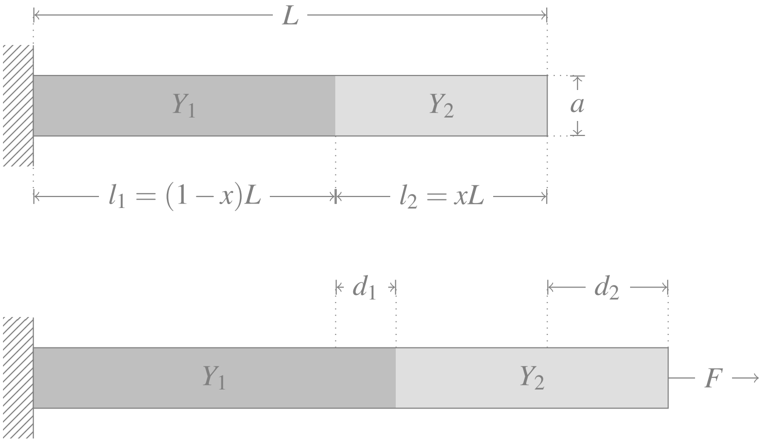
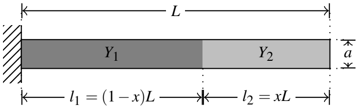
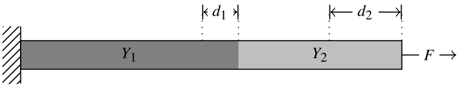


Goal Maximize x under the constraint that $d_2 < D$.

Two-component massless rod, tensile load: FE analysis





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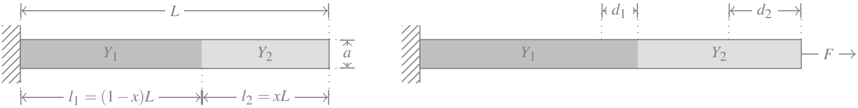
Two-component massless rod, tensile load: FE analysis

FE analysis 3 nodes, boundary conditions

$$\begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F \end{bmatrix}, \qquad c_i = \frac{Y_i a}{l_i}.$$



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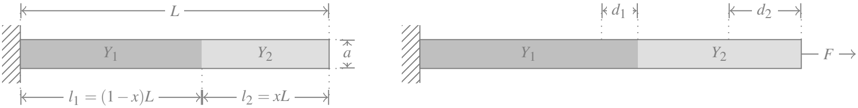

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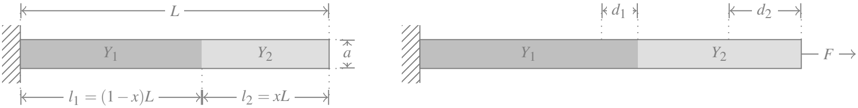

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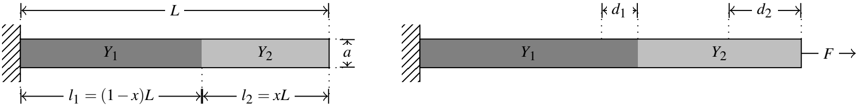
Two-component rod, tensile load: design optimization



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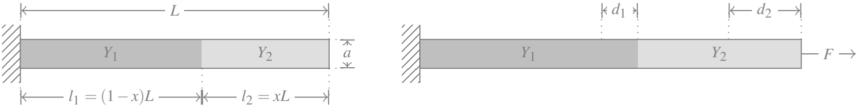
Two-component rod, tensile load: design optimization

Precisely known elastic moduli Y_1 and Y_2 This problem is

- a classical constrained optimization problem;
- considered ‘solved’.



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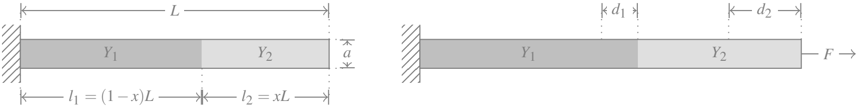

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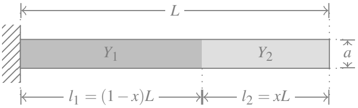
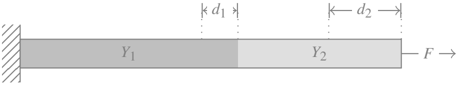
Approach:

- reformulate as a well-posed decision problem;
- solve the decision problem, i.e.,
  derive a classical constrained optimization problem.





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Overview

- Toy problem
- General problem formulation
- Uncertainty models
- Optimality criteria
- Probabilistic and indeterminacy aspects of uncertainty
- Objective
- Results
- Application: bridge design for vehicle-pillar collisions

A constrained optimization problem under uncertainty

Goal Maximize $f(x)$ under the constraint that xRY.

- x optimization variable (values in $\mathscr{X}$)
- f objective function (from $\mathscr{X}$ to $\mathbb{R}$)
- Y random variable (realizations y in $\mathscr{Y}$)
- R relation on $\mathscr{X} \times \mathscr{Y}$.

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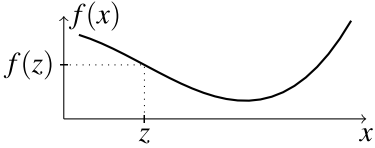
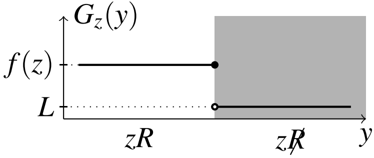
Decision problem Find the optimal decisions x:

- associate a utility function with every decision z:

$$G_z(y) = f(z)I_{zR} + LI_{z\not R} = \begin{cases} f(z), & zRy, \\ L, & z\not Ry, \end{cases}$$ with penalty value $L < \inf f$;





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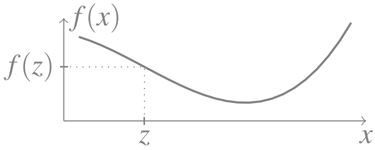

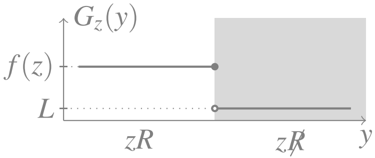

- choose an optimality criterion, e.g., maximinity, maximality.

Uncertainty models

Goal Faced with uncertainty about y in $\mathscr{Y}$, find optimal x in $\mathscr{X}$ given an optimality criterion and utility functions G_z on $\mathscr{Y}$ for all z in $\mathscr{X}$.

Uncertainty models

Random variable Y Formal model for the uncertainty about y in $\mathscr{Y}$.

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Lower and upper expectation With (almost) all typical uncertainty models correspond lower and upper expectation operators ($\underline{E}$ and $\overline{E}$), or (almost) equivalently, a set of linear expectation operators $\mathscr{M}$:

$$\underline{E}_{\mathscr{M}}(G) := \inf_{E \in \mathscr{M}} E(G), \qquad \overline{E}_{\mathscr{M}}(G) := \sup_{E \in \mathscr{M}} E(G),$$

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Examples

- probabilities (measures, PMF, PDF, CDF);
- upper and/or lower of the above (inner/outer measures, Choquet capacities, p-boxes);
- intervals, vacuous expectations: $\underline{E}_A(G) := \inf_{y\in A} G(y)$;
- possibility distributions, belief functions, ...
- convex mixtures of the lot (e.g., contamination models).

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Optimality criteria: maximizing expected utility generalized

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Maximinity Worst-case reasoning; optimal x maximize the lower (minimal) expected utility ($\underline{P}(A) := \underline{E}(I_A)$):

$$\begin{aligned}\underline{E}(G_x) &= \sup_{z\in\mathscr{X}} \underline{E}(G_z)\\ &= \sup_{z\in\mathscr{X}} \underline{E}\big(f(z)I_{zR} + LI_{z\not R}\big) = L + \sup_{z\in\mathscr{X}} \big(f(z) - L\big)\underline{P}(zR).\end{aligned}$$

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Maximality Optimal x are undominated in pairwise comparisons with all other decisions:

$$\begin{aligned}0 &\leq \inf_{z\in\mathscr{X}} \overline{E}(G_x - G_z)\\ &= \inf_{z\in\mathscr{X}} \overline{E}\Big(\big(f(x) - f(z)\big)I_{xR\cap zR} + \big(f(x) - L\big)I_{xR\cap z\not R} + \big(L - f(z)\big)I_{x\not R\cap zR}\Big).\end{aligned}$$

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Others Maximaxity, E-admissibility, interval dominance

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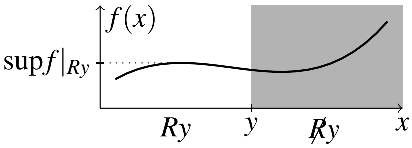
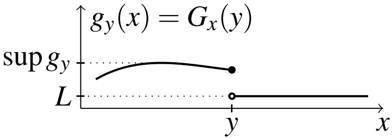
Probabilistic and indeterminacy aspects of uncertainty

Example $\mathscr{X} = \mathscr{Y} := \mathbb{R}$, $R := \leq$.





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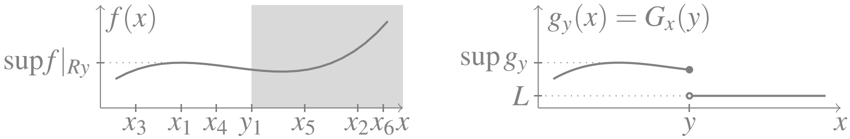
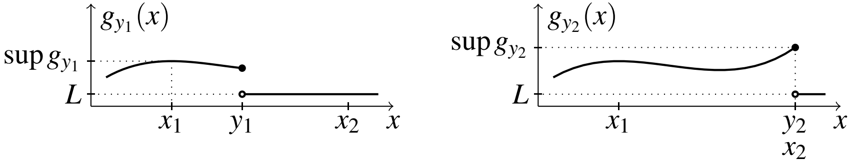
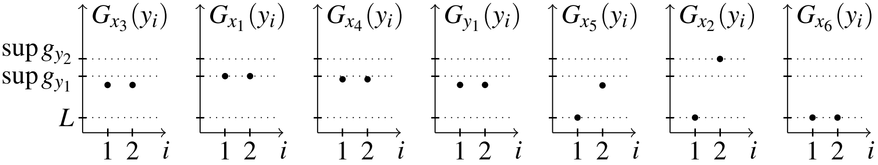


Indeterminacy Assume y can be either y_1 or y_2, but nothing more is known.





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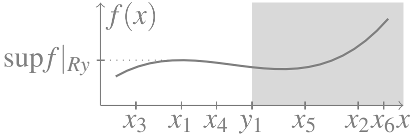
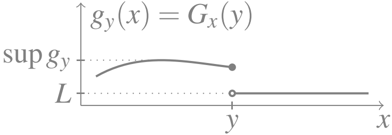
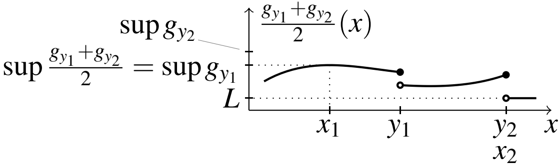
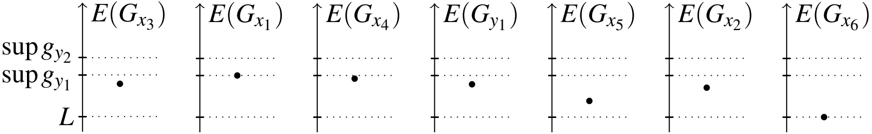




Probabilistic Assume that y_1 and y_2 are equally likely.





Objective, deliverables, and a disclaimer

Research objective decision problem solutions for combinations of various uncertainty models and optimality criteria.

Deliverables A solution toolbox for a specific, but quite general class of decision problems under uncertainty.

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Disclaimer No reduction in the computational complexity; one faces

- an optimization problem to find the uncertainty-independent constraints,
- the resulting classical constrained optimization problem.

Results: probabilities

Optimal decision when Y is described by a probability P.

Maximizing expected utility

- General case:

$$\operatorname{argsup}_{z\in\mathscr{X}}(f(z)-L)P(zR).$$

- Example: $\mathscr{X}=\mathscr{Y}:=\mathbb{R}$, $R:=\leq$.

$$\operatorname{argsup}_{z\in\mathbb{R}}\big(f(z)-L\big)\big(1-F(z)\big),$$

where $F_Y(x):=P(\mathbb{R}_{\leq x})=1-P(x\leq)$
is a continuous CDF.

Results: vacuous models

Optimal decision when Y is described by
a vacuous lower expectation relative to $A \subseteq \mathscr{Y}$.

Maximinity

- General case:

$$\operatorname{argsup}_{z \in \underline{RA}} f(z), \quad \underline{RA} := \bigcap_{y \in A} Ry.$$

- Example: $\mathscr{X} = \mathscr{Y} := \mathbb{R}$, $R := \leq$, $A := [a, b]$.

$$\operatorname{argsup}_{z \leq a} f(z).$$

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$$\operatorname{argsup}_{z \leq a} f(z).$$

Maximality

- General case:

$$x \in \overline{RA} \quad \text{such that} \quad f(x) = \sup_{z \in \underline{RA}} f(z), \quad \overline{RA} := \bigcup_{y \in A} Ry.$$

- Example: $\mathscr{X} = \mathscr{Y} := \mathbb{R}$, $R := \leq$, $A := [a,b]$.

$$x \leq b \quad \text{such that} \quad f(x) \geq \sup_{z \leq a} f(z).$$

Results: possibility distributions

Optimal decision when Y is described by
a possibility distribution π on $\mathscr{Y}$; $\underline{P}(A) := 1 - \sup_{y \in \mathscr{Y} \setminus A} \pi(y)$.

Maximinity

- General case:

$$\operatorname{argsup}_{z \in \mathscr{X}} \big(f(z) - L\big)\big(1 - \sup_{y \in z R} \pi(y)\big).$$

- Example: $\mathscr{X} = \mathscr{Y} := \mathbb{R}$, $R := \leq$, continuous π with minimal mode $c \in \mathbb{R}$.

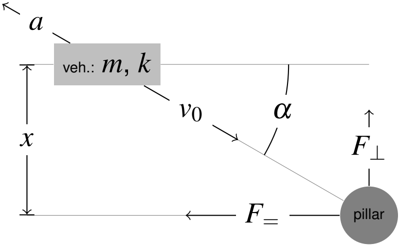
$$\operatorname{argsup}_{z < c} \big(f(z) - L\big)\big(1 - \pi(z)\big).$$

Bridge design: vehicle colliding into pillar

Vehicle parameters mass m, stiffness k, initial speed v_0, average deceleration a, and swerve angle α.

Bridge parameters pillar design loads $F_=$ (longitud.) and $F_\perp$ (perpendic.).

What is the optimal lateral distance x between the vehicle and curb that ensures structural integrity?



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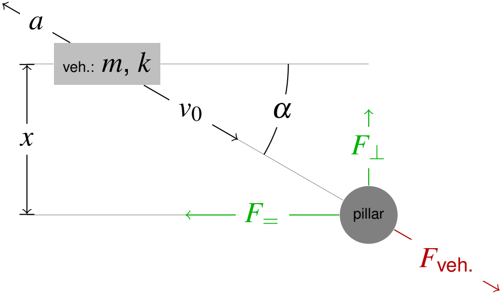
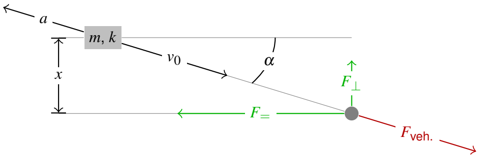


Structural integrity constraint

$$F_{\text{veh.}} \cos\alpha \leq F_=, \quad F_{\text{veh.}} \sin\alpha \leq F_\perp, \quad F_{\text{veh.}} = \sqrt{mk(v_0^2 - 2ax/\sin\alpha)}.$$

Bridge design: optimization problem under uncertainty

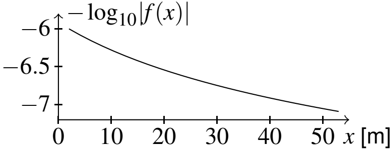
Goal Choose an optimal x under the constraint that $F_{\text{veh.}} \cos\alpha \leq F_{=}$ and $F_{\text{veh.}} \sin\alpha \leq F_{\perp}$.



Bridge design: optimization problem under uncertainty

Objective function Based on dimensions-dependent building costs:

$$f(x) := -45B\left((L_1 + 2d)^2 + 2L_2^2\right),$$



where $B = 14$, $L_1 = 33$, $L_2 = 15$ for a typical 3-span bridge.

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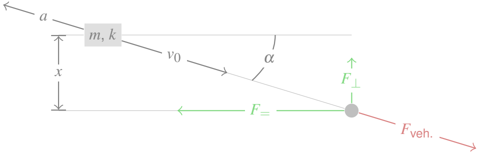

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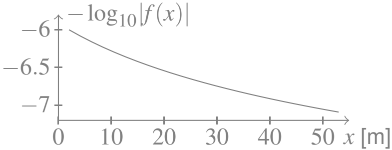
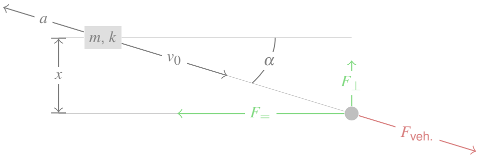


where $B = 14$, $L_1 = 33$, $L_2 = 15$ for a typical 3-span bridge.

Penalty value L was difficult to assess, so a number of values between $-10^{6.3}$ and $-10^{8.6}$ were tried.

Parameters $k = 300$ [kN/m]; $Y = (m, v_0, a, \alpha)$, independent product.

Goal Choose an optimal x under the constraint that $F_{\text{veh.}} \cos\alpha \leq F_{=}$ and $F_{\text{veh.}} \sin\alpha \leq F_{\perp}$.



Bridge design: uncertainty models for the parameters

Mass m [t]

- Lorry: normal with mean 20, standard deviation 12, and realistic range $[12, 40]$.
- Car: vacuous in the interval $[.5, 1.6]$.

Initial velocity v_0 [km/h]

- Highway: 80, 10, $[50, 100]$.
- Urban: 40, 8, $[30, 70]$.
- Courtyard: lognormal 15, 5, $[5, 30]$.
- Parking: lognormal 5, 5, $[5, 20]$.

Average deceleration a [m/s^2] Lognormal 4, 1.3, $[1, 5]$.

Swerve angle α [°] Normal 30, 3, $[8, 45]$.

Bridge design: maximinity results for different vehicle types

Lorry – Highway $F_{=} = 1000$, $F_{\perp} = 500$

Lorry – Urban $F_{=} = 500$, $F_{\perp} = 250$

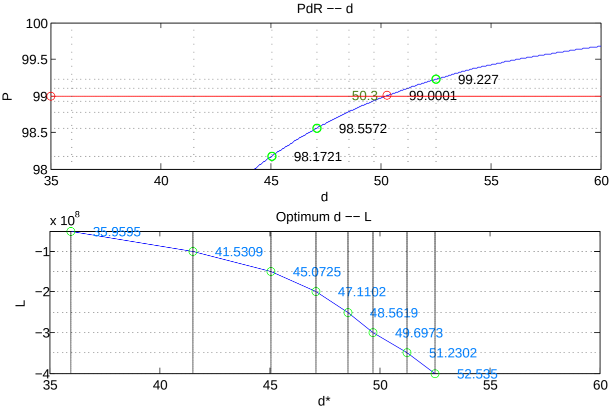
Lorry – Courtyard $F_{=} = 150$, $F_{\perp} = 75$

Car – Courtyard $F_{=} = 50$, $F_{\perp} = 25$

Car – Parking $F_{=} = 40$, $F_{\perp} = 25$

Bridge design: maximinity results for different vehicle types

Lorry – Highway $F_= = 1000, F_\perp = 500$



Lorry – Urban $F_= = 500, F_\perp = 250$

Lorry – Courtyard $F_= = 150, F_\perp = 75$

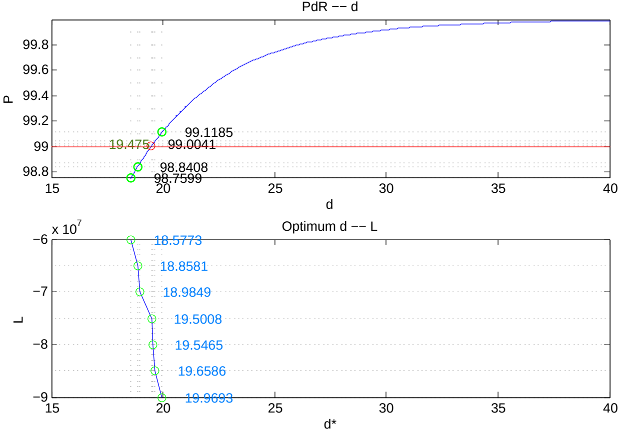
Car – Courtyard $F_= = 50, F_\perp = 25$

Car – Parking $F_= = 40, F_\perp = 25$

Bridge design: maximinity results for different vehicle types

Lorry – Highway $F_= = 1000$, $F_\perp = 500$, $x = 42$ for $L = -10^8$.

Lorry – Urban $F_= = 500$, $F_\perp = 250$



Lorry – Courtyard $F_= = 150$, $F_\perp = 75$

Car – Courtyard $F_= = 50$, $F_\perp = 25$

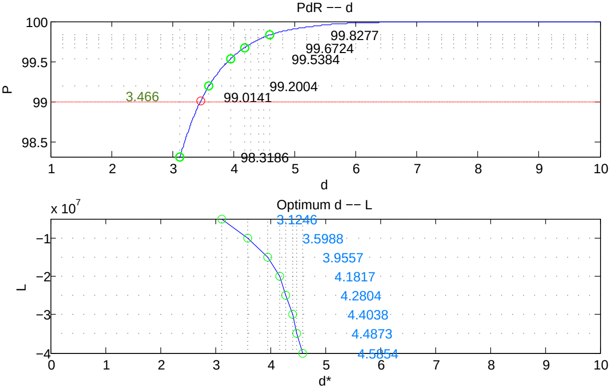
Car – Parking $F_= = 40$, $F_\perp = 25$

Bridge design: maximinity results for different vehicle types

Lorry – Highway $F_{=} = 1000$, $F_{\perp} = 500$, $x = 42$ for $L = -10^8$.

Lorry – Urban $F_{=} = 500$, $F_{\perp} = 250$, $x = 20$ for $L = -10^8$.

Lorry – Courtyard $F_{=} = 150$, $F_{\perp} = 75$



Car – Courtyard $F_{=} = 50$, $F_{\perp} = 25$

Car – Parking $F_{=} = 40$, $F_{\perp} = 25$

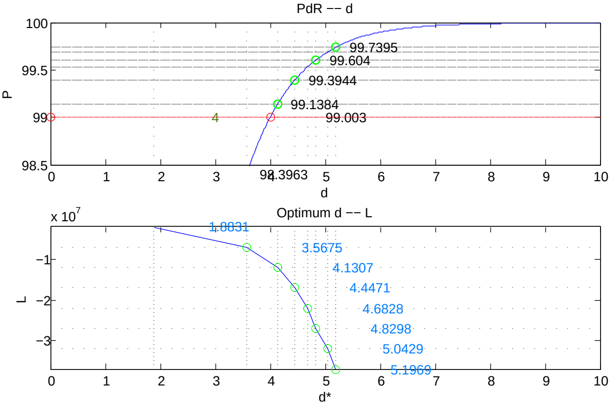
Bridge design: maximinity results for different vehicle types

Lorry – Highway $F_= = 1000$, $F_\perp = 500$, $x = 42$ for $L = -10^8$.

Lorry – Urban $F_= = 500$, $F_\perp = 250$, $x = 20$ for $L = -10^8$.

Lorry – Courtyard $F_= = 150$, $F_\perp = 75$, $x = 3.6$ for $L = -10^7$.

Car – Courtyard $F_= = 50$, $F_\perp = 25$



Car – Parking $F_= = 40$, $F_\perp = 25$

Bridge design: maximinity results for different vehicle types

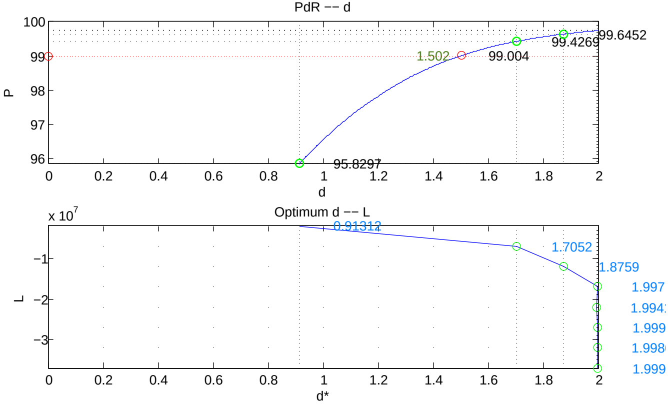
Lorry – Highway $F_{=} = 1000$, $F_{\perp} = 500$, $x = 42$ for $L = -10^8$.

Lorry – Urban $F_{=} = 500$, $F_{\perp} = 250$, $x = 20$ for $L = -10^8$.

Lorry – Courtyard $F_{=} = 150$, $F_{\perp} = 75$, $x = 3.6$ for $L = -10^7$.

Car – Courtyard $F_{=} = 50$, $F_{\perp} = 25$, $x = 4.0$ for $L = -10^7$.

Car – Parking $F_{=} = 40$, $F_{\perp} = 25$



Bridge design: maximinity results for different vehicle types

Lorry – Highway $F_{=} = 1000$, $F_{\perp} = 500$, $x = 42$ for $L = -10^8$.

Lorry – Urban $F_{=} = 500$, $F_{\perp} = 250$, $x = 20$ for $L = -10^8$.

Lorry – Courtyard $F_{=} = 150$, $F_{\perp} = 75$, $x = 3.6$ for $L = -10^7$.

Car – Courtyard $F_{=} = 50$, $F_{\perp} = 25$, $x = 4.0$ for $L = -10^7$.

Car – Parking $F_{=} = 40$, $F_{\perp} = 25$, $x = 1.8$ for $L = -10^7$.

Bridge design: maximality results for different vehicle types

Car – Courtyard $F_{=} = 50$, $F_{\perp} = 25$, $x \in [2.4, 4.0]$ for $L = -12 \cdot 10^6$.

Car – Parking $F_{=} = 40$, $F_{\perp} = 25$, $x \in [0.4, 1.7]$ for $L = -7 \cdot 10^6$.