# The determinant of a parameterized matrix

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*Problem statement* Let us consider the following problem: Find the expression of the determinant of a real-valued square matrix $A$ of dimension $n$ in $\mathbb{N}$,[1] of which some components contain polynomial functions of $m$ parameters $a_i$, where $i \leq m$ and $m$ is some natural number.

*Determinant expression* The determinant function is multilinear in the components of its matrix argument, so $\det A$ will be polynomial in the parameters $a_i$. This means we can write this expression as $\sum_{\boldsymbol{p} \in P} c_{\boldsymbol{p}} \boldsymbol{a}^{\boldsymbol{p}}$, where the $c_{\boldsymbol{p}}$ are real coefficients, where $\boldsymbol{a}^{\boldsymbol{p}}$ is a shorthand for $\prod_{0<i\leq m} a_i^{p_i}$, and where $P$ is the set of tuples of powers $p_i$ for $a_i$ for all monomials appearing in the polynomial.

Determining $P$ exactly might be hard. We can, however, replace $P$ by an easily determinable encompassing set, such as $\bigtimes_{0<i\leq m} \mathbb{N}_{\leq \hat{p}_i}$, where $\hat{p}_i$ is the maximum power of $a_i$ occurring in any monomial; this will just result in more coefficients to be determined (to be zero). Similarly, each $\hat{p}_i$ can be replaced by an upper bound consisting of the product over the matrix components of all powers of $a_i$; again this will just result in more coefficients to be determined (to be zero). An analytical expressions for tighter encompassing sets would be welcome.

*Finding the coefficients* We can find the coefficients $c_{\boldsymbol{p}}$ as follows: By choosing $|P|$ distinct parameter vectors $\boldsymbol{a}_j \in \mathbb{R}^m$ we can create a linear system of $|P|$ equations $\det A(\boldsymbol{a}_j) = \sum_{\boldsymbol{p}\in P} c_{\boldsymbol{p}} {\boldsymbol{a}_j}^{\boldsymbol{p}}$. The $c_{\boldsymbol{p}}$ are then obtained by solving this system with an appropriate algorithm.

The liberty we have in choosing the $\boldsymbol{a}_j$ could be exploited to make the linear system as simple as possible to solve (e.g., by first exhausting the $\{0,1\}$-valued $\boldsymbol{a}_j$, then the $\{-1,0\}$-valued ones, etc.).

[1] We here follow the convention that 0 is not a natural number, i.e., $\mathbb{N} = \mathbb{Z}_{>0}$.

*Alternative factorization* Based on our knowledge of the structure of the matrix $A$ (sparsity, tridiagonality, etc.) and thus of $P$, it might be computationally advantageous to factorize $\det A$ in a specific way. For example, one could write it as $f_{\boldsymbol{a},\hat{\boldsymbol{p}}}(\boldsymbol{d})$ using the parameterized function $f$ introduced below, where $\boldsymbol{d}$ is an $\bigtimes_{0<i\leq|\boldsymbol{a}|}(\hat{p}_i+1)$-array of coefficients.

Let $\boldsymbol{\alpha}$ be a real vector of length $|\boldsymbol{\alpha}|$ and $\boldsymbol{\ell}$ in $\mathbb{N}^{|\boldsymbol{\alpha}|}$, then the function $f_{\boldsymbol{\alpha},\boldsymbol{\ell}}$ is defined recursively by

$$\begin{aligned} f_{\boldsymbol{\alpha},\boldsymbol{\ell}} &: \mathbb{R}^{\ell_1+1} \to \mathbb{R} \\ &: \boldsymbol{d} \mapsto \textstyle\sum_{0\leq k\leq \ell_1} d_k \alpha_1^k, && |\boldsymbol{\alpha}| = 1, \\ f_{\boldsymbol{\alpha},\boldsymbol{\ell}} &: \mathbb{R}^{\ell_{|\boldsymbol{\alpha}|}+1} \times \textstyle\bigtimes_{0<i<|\boldsymbol{\alpha}|} \mathbb{R}^{\ell_i+1} \to \mathbb{R} \\ &: \boldsymbol{d} \mapsto \textstyle\sum_{0\leq k\leq \ell_{|\boldsymbol{\alpha}|}} f_{\check{\boldsymbol{\alpha}},\check{\boldsymbol{\ell}}}(\boldsymbol{d}_k)\alpha_{|\boldsymbol{\alpha}|}^k, && |\boldsymbol{\alpha}| > 1, \end{aligned}$$

where $\check{\boldsymbol{r}}$ is the vector formed by dropping the last component of the vector $\boldsymbol{r}$. Note that a different factorization is obtained for all different orderings of the vector of parameters $\boldsymbol{a}$.

Now, how do we find the coefficients $d_{(k_i|0<i\leq|\boldsymbol{a}|)}$, with $0 \leq k_i \leq \hat{p}_i$, for this alternative factorization? Well, we start from the fact that $\det A(\boldsymbol{a}_j) = f_{\boldsymbol{a}_j,\hat{\boldsymbol{p}}}(\boldsymbol{d})$, which – when taking sets of $\hat{p}_i + 1$ distinct $a_i$-values and choosing the $\boldsymbol{a}_j$ out of their cartesian product – gives an implicit specification of $\boldsymbol{d}$ through $f_{\boldsymbol{a}_j,\hat{\boldsymbol{p}}}$. Because of the recursive definition of this function, we do not have to solve one large $|P|$-dimensional linear system to find the coefficients, but we now have to solve a large number of small ($\hat{p}_i + 1$-dimensional) linear systems.

*Application: parameterized linear systems* Given a linear system defined by $b$ and $A$, a real-valued vector and square matrix of dimension $n$, respectively, containing polynomial parametric expressions in some components. Cramer's rule reduces solving linear systems to calculating determinants; therefore, using the ideas of this note, we can obtain solution expressions under the form of parametric rational functions.