

# Exchangeability for sets of desirable gambles

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# Immediate prediction under exchangeability . . .

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**Immediate prediction under exchangeability and representation insensitivity**

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**Immediate prediction under exchangeability & representation insensitivity**  
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**Representation insensitivity in immediate prediction under exchangeability<sup>a</sup>**

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**ABSTRACT**

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**1. Introduction**

Consider a subject who is making  $N > 0$  successive observations of a certain phenomenon. We represent these observations by  $N$  random variables  $X_1, \dots, X_N$ . The random variable  $X_n$  represents a variable whose value the subject may estimate and predict. We assume that at each observation instant  $n$ , the actual value of the random variable  $X_n$  can be determined in principle. To be able, one subject might be looking for bugs in the Russian forest, and then it is the operator of the lifting for mines around the, for example, an archaeological excavation. For training both without replacement from one set, as which case  $X_n$  could designate the color of the UK ball labels from the urn.

**2. Overview**

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# Exchangeability for sets of desirable gambles

Desirability

## General context: experiments & gambles

A from possibility space  $\Omega$  of outcomes of some experiment.  
A subject sets a collection about the experiment's outcome.  
Satisfied  $f \in \mathcal{F}(\Omega) := \mathbb{R}^\Omega$ , interpreted as gambles means:  
 $f(\omega)$  when the experiment's outcome is  $\omega$ .



A gamble  $f$  is desirable to the subject if it satisfies the following three properties:  
(1) Its actual outcome is  $\geq 0$ , and  
(2) Its subjectively expected value is  $\geq 0$ .  
The same gamble  $f$  is not desirable.

## Coherent sets of desirable gambles

A subject's set of desirable gambles  $\mathcal{D} \subseteq \mathcal{F}(\Omega)$  models his beliefs about the experiment's outcome.

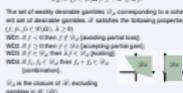


The set of desirable gambles  $\mathcal{D}$  is coherent if it satisfies the following stability requirements:  
(1)  $\mathcal{D} \subseteq \mathcal{F}(\Omega)$ ,  $\mathcal{D} \neq \emptyset$ .  
(2)  $f, g \in \mathcal{D} \Rightarrow f + g \in \mathcal{D}$ .  
(3)  $f \in \mathcal{D} \Rightarrow f + \lambda \mathbf{1} \in \mathcal{D}$ ,  $\lambda > 0$ .  
(4)  $f \in \mathcal{D} \Rightarrow f + \lambda \mathbf{1} \in \mathcal{D}$ ,  $\lambda > 0$ .  
(5)  $f, g \in \mathcal{D} \Rightarrow f + \lambda \mathbf{1} \in \mathcal{D}$ ,  $\lambda > 0$ .  
(6)  $f, g \in \mathcal{D} \Rightarrow f + \lambda \mathbf{1} \in \mathcal{D}$ ,  $\lambda > 0$ .  
Requirements (2) and (3) make  $\mathcal{D}$  a convex cone of  $\mathcal{F}(\Omega)$ .

## Sets of weakly desirable gambles

The subject considers a gamble  $f \in \mathcal{F}(\Omega)$  weakly desirable if he would not accept  $f$  if he had to choose between  $f$  and another desirable gamble  $g$ .

The subject's set of weakly desirable gambles is  $\mathcal{D}_w := \{f \in \mathcal{F}(\Omega) : f \neq -g, g \in \mathcal{D}\}$ .



## Assessments & their natural extension

An assessment can consist of a set  $\mathcal{A} \subseteq \mathcal{F}(\Omega)$  considered desirable by the subject.

The assessment of  $\mathcal{A}$  is not possibly under the reduction of  $\text{red}(\mathcal{A}, \mathcal{D})$  and  $\mathcal{D}$  is empty.

The natural extension  $\mathcal{D}^e$  of  $\mathcal{A}$  is

$$\mathcal{D}^e := \text{red}(\mathcal{A} \cup \mathcal{D}, \mathcal{D})$$

If  $\mathcal{A}$  is assessably non-possibly, then  $\mathcal{D}^e$  is the smallest coherent set of desirable gambles including  $\mathcal{A}$ .

## Updating sets of desirable gambles

The subject observes, or considers the possibility of observing, an event  $B \subseteq \Omega$ .

Contingent on observing  $B$ , the subject models his beliefs using an updated set of desirable gambles, the subject's  $\mathcal{D}^B$  given by

$$\mathcal{D}^B := \{f \in \mathcal{F}(\Omega) : f \neq -g, g \in \mathcal{D}\}$$

If  $\mathcal{D}$  is a coherent set of desirable gambles on  $\Omega$ , then  $\mathcal{D}^B$  is a coherent set of desirable gambles on  $B$ .

## Coherent previsions & desirability

The lower prevision of a gamble  $f$  established by a set of desirable gambles  $\mathcal{D}$  is

$$\underline{P}(f) := \inf\{c \in \mathbb{R} : f - c \mathbf{1} \in \mathcal{D}\}$$

Its conjugate upper prevision  $\overline{P}(f)$  is

$$\overline{P}(f) := \sup\{c \in \mathbb{R} : c - f \in \mathcal{D}\}$$

A lower prevision  $\underline{P}$  is coherent if there exists some coherent set of desirable gambles  $\mathcal{D}$  such that  $\underline{P}(f) = \inf\{c \in \mathbb{R} : f - c \mathbf{1} \in \mathcal{D}\}$ .

Coherent lower previsions are best representable uncertainty models, their coherent sets of desirable gambles.

Exchangeability

## Specific context: finite sequences

The experiment consists of the observation of the value of a sequence  $X_1, \dots, X_n$  of random variables for which  $\Omega$  is the finite set of possible values. So the possibility space is  $\mathcal{F} := \{f : \Omega \rightarrow \mathbb{R}\}$ .

A gamble  $f$  is  $\mathcal{D}$  and  $\mathcal{D}_w$  respectively.

$\mathcal{D}_w$  is the set of all permutations  $\pi$  of the index set  $\{1, \dots, n\}$ .

The associated permutation  $\pi^*$  is defined by  $\pi^*(f) := f \circ \pi$ .

A set  $\mathcal{D}$  is a permutation of  $\mathcal{D}$  if  $\mathcal{D} = \{\pi^*(f) : f \in \mathcal{D}\}$ .

With every sequence of observations corresponding a joint vector in  $\mathbb{R}^n := \{x \in \mathbb{R}^n : x_i \in \Omega_i, i=1, \dots, n\}$ .

The desirability  $\mathcal{D}$  is  $\mathcal{D}$  if  $\mathcal{D} = \{f \in \mathcal{F}(\Omega) : f \geq 0\}$ .

Permitted gambles have the same joint vector  $x$  a permutation of  $x$ .

$\mathcal{D} = \{f \in \mathcal{F}(\Omega) : f \geq 0\}$ .

$\mathcal{D}_w = \{f \in \mathcal{F}(\Omega) : f \geq 0\}$ .

## Exchangeability

A subject assesses that  $X_1, \dots, X_n$  are exchangeable. This means that for any gamble  $f$  and permutation  $\pi$ , he finds exchangeable  $f$  for  $\pi^*(f)$  for  $\mathcal{D}$  or  $\mathcal{D}_w$  respectively.

The negative invariance property of each such exchangeable set is

$$\mathcal{D}_w = \{f \in \mathcal{F}(\Omega) : f \geq 0\}$$

A  $\mathcal{D}_w$  assesses weakly desirable gambles, then so does its conical hull  $\mathcal{D}_w^c := \text{red}(\mathcal{D}_w, \mathcal{D}_w)$ .

A subset  $\mathcal{A}$  of all assessment gambles on  $\Omega^n$  is called exchangeable if  $\mathcal{A} \subseteq \mathcal{D}_w^c$  or equivalently  $\mathcal{A} \subseteq \mathcal{D}_w^c$ .

If  $\mathcal{A}$  is coherent and exchangeable then it is also permutation for all  $\pi$  and if  $\mathcal{A}$  and  $\mathcal{D}$  holds that  $\mathcal{A} \subseteq \mathcal{D}$ .

## Exchangeable natural extension

The assessment of  $\mathcal{A}$  is not possibly under exchangeability if  $\mathcal{A} \subseteq \mathcal{D}_w^c$  avoids non-possibility.

The exchangeable natural extension  $\mathcal{D}^e$  of  $\mathcal{A}$  is

$$\mathcal{D}^e := \text{red}(\mathcal{A} \cup \mathcal{D}, \mathcal{D})$$

If  $\mathcal{A}$  is assessably non-possibly under exchangeability then  $\mathcal{D}^e$  is the smallest exchangeable coherent set of desirable gambles including  $\mathcal{A}$ .

## Updating exchangeable models

The subject observes the values  $x = (x_1, \dots, x_n)$  at the joint vector in  $\mathbb{R}^n$  or the first  $j$  variables  $X_1, \dots, X_j$  the event observing the event  $\{j\} \subseteq \Omega^n := \mathcal{B}^j$ . We are interested in observations about the relationship  $x \in \mathcal{B}^j$  or not.

Contingent on observing  $j$  or  $\mathcal{B}^j$ , the subject models his beliefs using updated sets of desirable gambles, the subject's  $\mathcal{D}^{\mathcal{B}^j}$  that are

$$\mathcal{D}^{\mathcal{B}^j} := \{f \in \mathcal{F}(\Omega) : f \geq 0\}$$

If  $\mathcal{D}$  is a coherent and exchangeable set of desirable gambles on  $\Omega^n$ , then  $\mathcal{D}^{\mathcal{B}^j}$  is a coherent and exchangeable set of desirable gambles on  $\mathcal{B}^j$ .

On  $\mathcal{B}^j$ , the subject's set of desirable gambles is  $\mathcal{D}^{\mathcal{B}^j}$ .

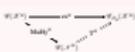
On  $\mathcal{B}^j$ , the subject's set of desirable gambles is  $\mathcal{D}^{\mathcal{B}^j}$ .

## Exchangeable previsions

A lower prevision  $\underline{P}$  on  $\mathcal{F}(\Omega^n)$  is exchangeable if there is some exchangeable coherent set of desirable gambles  $\mathcal{D}$  such that  $\underline{P}(f) = \inf\{c \in \mathbb{R} : f - c \mathbf{1} \in \mathcal{D}\}$ .

Representation

## Moving between sequence gambles and count gambles



The set of permutation-invariant sequence gambles is  $\mathcal{F}(S^n) := \{f \in \mathcal{F}(S^n) : f \circ \pi = f\}$ .

The projection of a sequence gamble  $f$  onto a permutation-invariant gamble is

$$\text{red}(f, \mathcal{F}(S^n)) := \sum_{\pi \in \mathcal{P}(S^n)} \text{red}(f \circ \pi, \mathcal{F}(S^n))$$

whose value at an individual  $\omega$  is given by

$$\text{red}(f, \mathcal{F}(S^n))(\omega) := \sum_{\pi \in \mathcal{P}(S^n)} f(\pi(\omega))$$

The count gamble corresponding to the sequence gamble  $f$  is  $\text{red}(f, \mathcal{F}(S^n))$ .

The permutation-invariant sequence gamble is a one-to-one correspondence with the count gamble  $\mathcal{F}(S)$ .

$$\text{red}(f, \mathcal{F}(S^n)) = \text{red}(f, \mathcal{F}(S^n))$$

## Representation

A set of desirable gambles  $\mathcal{D}$  on  $\Omega^n$  is coherent and exchangeable if there is some coherent set  $\mathcal{D}$  of desirable gambles on  $\mathcal{F}^n := \mathcal{F}(S^n)$  such that

$$\mathcal{D} = \{\text{red}(f, \mathcal{F}(S^n)) : f \in \mathcal{D}\}$$

and in that case this  $\mathcal{D}$  is uniquely determined by

$$\mathcal{D} = \{\text{red}(f, \mathcal{F}(S^n)) : f \in \mathcal{D}\}$$

## Representing updated models

The subject observes the values  $x = (x_1, \dots, x_n)$  at the joint vector in  $\mathbb{R}^n$  or the first  $j$  variables  $X_1, \dots, X_j$ .

If  $\mathcal{D}$  is a coherent and exchangeable set of desirable gambles on  $\Omega^n$ , then the representation of the lower prevision of exchangeable updated models is given by

$$\underline{P}^{\mathcal{B}^j}(f) := \inf\{c \in \mathbb{R} : f - c \mathbf{1} \in \mathcal{D}^{\mathcal{B}^j}\}$$

This representation is not an updated model of the representation  $\underline{P} := \inf\{c \in \mathbb{R} : f - c \mathbf{1} \in \mathcal{D}\}$  of  $\mathcal{D}$ . They are however related by

$$\underline{P}^{\mathcal{B}^j}(f) = \inf\{c \in \mathbb{R} : f - c \mathbf{1} \in \mathcal{D}^{\mathcal{B}^j}\}$$

when we use the dual functional, defined for every count vector  $\omega$  in  $\mathbb{R}^n$  by

$$\underline{P}(\omega) := \inf\{c \in \mathbb{R} : c \mathbf{1} - \omega \in \mathcal{D}\}$$

which is zero when  $\omega \in \mathcal{B}^j$ .

## Exchangeable previsions & representation

A lower prevision  $\underline{P}$  on  $\mathcal{F}(S^n)$  is coherent and exchangeable if there is some coherent lower prevision  $\underline{P}$  on  $\mathcal{F}(S)$  such that

$$\underline{P}(f) = \inf\{c \in \mathbb{R} : f - c \mathbf{1} \in \mathcal{D}\}$$

and in that case this  $\underline{P}$  is uniquely determined by

$$\underline{P}(f) = \inf\{c \in \mathbb{R} : f - c \mathbf{1} \in \mathcal{D}\}$$

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$$\underline{P}(f) = \inf\{c \in \mathbb{R} : f - c \mathbf{1} \in \mathcal{D}\}$$

- ▶ exchangeability assessment & weakly desirable gambles
- ▶ exchangeable natural extension
- ▶ updating exchangeable models
- ▶ relationship with exchangeable previsions

# Exchangeability for sets of desirable gambles

Desirability

## General context: experiments & gambles

A from possibility space  $\Omega$  of outcomes of some experiment.  
A subject sets a preference about the experiment's outcome.  
Standard  $f: \Omega \rightarrow \mathbb{R}^n$  interpreted as gambles means:  
 $f(\omega)$  is the experiment's outcome is  $\omega$ .



- A gamble  $f$  is desirable to the subject if it satisfies the following three properties:
  - The actual outcome is  $\omega$  determined, and
  - The subject's capital is changed by  $f(\omega)$ .
  - The new gamble is not desirable.

## Coherent sets of desirable gambles

A subject's set of desirable gambles  $\mathcal{D} \subseteq \mathbb{R}^n$  models his beliefs about the experiment's outcome.

The set of desirable gambles  $\mathcal{D}$  is coherent if it satisfies the following rationality requirements:

- $\mathcal{D}$  is a convex cone.
- $\mathcal{D} \cap (-\mathcal{D}) = \{0\}$ .
- $\mathcal{D} \cap \mathbb{R}^+ = \{0\}$ .
- $\mathcal{D} \cap \mathbb{R}^- = \{0\}$ .
- $\mathcal{D} \cap \mathbb{R}^+ = \{0\}$ .
- $\mathcal{D} \cap \mathbb{R}^- = \{0\}$ .
- $\mathcal{D} \cap \mathbb{R}^+ = \{0\}$ .
- $\mathcal{D} \cap \mathbb{R}^- = \{0\}$ .
- $\mathcal{D} \cap \mathbb{R}^+ = \{0\}$ .
- $\mathcal{D} \cap \mathbb{R}^- = \{0\}$ .



## Sets of weakly desirable gambles

The subject considers a gamble  $f \in \mathbb{R}^n$  weakly desirable if he holds any desirable gamble  $g$  & another desirable gamble  $h$  observed, so  $f \geq g$  if then  $f \geq h$ .

The subject's set of weakly desirable gambles is

$$\mathcal{D}_w = \{f \in \mathbb{R}^n \mid f \geq g, g \in \mathcal{D}\}$$

The set of weakly desirable gambles  $\mathcal{D}_w$  corresponding to a coherent set of desirable gambles  $\mathcal{D}$  satisfies the following properties:

- $\mathcal{D}_w \cap (-\mathcal{D}_w) = \{0\}$ .
- $\mathcal{D}_w \cap \mathbb{R}^+ = \{0\}$ .
- $\mathcal{D}_w \cap \mathbb{R}^- = \{0\}$ .
- $\mathcal{D}_w \cap \mathbb{R}^+ = \{0\}$ .
- $\mathcal{D}_w \cap \mathbb{R}^- = \{0\}$ .
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- $\mathcal{D}_w \cap \mathbb{R}^+ = \{0\}$ .
- $\mathcal{D}_w \cap \mathbb{R}^- = \{0\}$ .



## Assessments & their natural extension

An assessment can consist of a set  $\mathcal{A} \subseteq \mathbb{R}^n$  (considered desirable) by the subject.

The assessment of a gamble  $f$  under non-possibility of the interaction of  $\text{conv}(\mathcal{A})$  &  $\mathcal{D}_w$  is empty.

The natural extension of  $\mathcal{A}$  is

$$\mathcal{D}_w(\mathcal{A}) := \text{conv}(\mathcal{A}) \cap \mathcal{D}_w$$

If  $\mathcal{A}$  is acyclic non-possibility, then  $\mathcal{D}_w(\mathcal{A})$  is the smallest coherent set of desirable gambles including  $\mathcal{A}$ .

## Updating sets of desirable gambles

The subject observes, or considers the possibility of observing, an event  $B \subseteq \Omega$ .

Contingent on observing  $B$ , the subject modifies his beliefs using an updated set of desirable gambles. The subject's  $\mathcal{D}(B)$  given by

$$\mathcal{D}(B) := \{g \in \mathbb{R}^n \mid g \geq 0\}$$

If  $\mathcal{D}$  is a coherent set of desirable gambles on  $\Omega$ , then  $\mathcal{D}(B)$  is a coherent set of desirable gambles on  $B$ .

## Coherent previsions & desirability

The lower prevision of a gamble  $f$  established to a set of desirable gambles  $\mathcal{D}$  is

$$\underline{E}(f) := \sup\{c \in \mathbb{R} \mid f - c \in \mathcal{D}\}$$

Its conjugate upper prevision  $\overline{E}(f)$  is

$$\overline{E}(f) := \inf\{c \in \mathbb{R} \mid f - c \in -\mathcal{D}\}$$

A lower prevision  $\underline{E}$  is coherent if there exists a coherent set of desirable gambles  $\mathcal{D}$  such that  $\underline{E}(f) = \overline{E}(f)$ .

Coherent lower previsions are less expressive uncertainty models than coherent sets of desirable gambles.



Exchangeability

## Specific context: finite sequences

The experiment consists of the observation of the value of a sequence  $\omega_1, \dots, \omega_n$  of random variables to which  $\Omega$  is the finite set of possible values. So the possibility space is  $\Omega = \mathbb{R}^n$  and  $\mathcal{D} = \{0\}$ .

$\mathcal{D}_w$  is the set of all permutations  $\pi$  of the sides set  $\{1, \dots, n\}$ .

The associated permutation of  $\mathcal{D}_w$  is defined by  $(\mathcal{D}_w)_\pi = \mathcal{D}_w$ .

With every sequence of observations corresponds a joint vector in  $\mathbb{R}^n$ .

The corresponding  $\mathcal{D}_w$  is  $\mathcal{D}_w = \{0\}$ .

The identification  $\mathcal{D}_w \rightarrow \mathcal{D}_w$  maps  $\{0\} \rightarrow \{0\}$ .

Permitted sequences have the same joint vector: a permutation invariant gamble.

$\mathcal{D}_w = \{0\}$ .

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## Exchangeability

A subject assesses that  $\omega_1, \dots, \omega_n$  are exchangeable. This means that for any gamble  $f$  and any permutation  $\pi$ , he holds exchanging  $f$  by  $f \circ \pi$  weakly desirable, because he is indifferent between them. The negative natural extension of each such exchange gamble is

$$\mathcal{D}_w(\mathcal{A}) := \{f - f \circ \pi \mid f \in \mathbb{R}^n, \pi \in \mathcal{D}_w\}$$

If  $\mathcal{D}_w$  consists of weakly desirable gambles, then so does its conical hull  $\mathcal{D}_w(\mathcal{A})$ .

A subset  $\mathcal{A}$  of all exchange gambles on  $\mathbb{R}^n$  is called exchangeable if  $\mathcal{D}_w \subseteq \mathcal{D}_w(\mathcal{A})$  or equivalently if

$$\mathcal{D}_w \subseteq \mathcal{D}_w(\mathcal{A})$$

If  $\mathcal{A}$  is coherent and exchangeable then it is also permutation for  $\mathcal{A} \cap \mathcal{D}_w$  and all  $\pi \in \mathcal{D}_w$  holds that  $\mathcal{A} \cap \mathcal{D}_w = \mathcal{A} \cap \mathcal{D}_w \circ \pi$ .

## Exchangeable natural extension

The assessment of a gamble  $f$  under non-possibility under exchangeability of  $\mathcal{A} \cap \mathcal{D}_w$  avoids non-possibility.

The exchangeable natural extension of  $\mathcal{A}$  is

$$\mathcal{D}_w(\mathcal{A}) := \text{conv}(\mathcal{A}) \cap \mathcal{D}_w$$

If  $\mathcal{A}$  is acyclic non-possibility under exchangeability, then  $\mathcal{D}_w(\mathcal{A})$  is the smallest non-possibility coherent set of desirable gambles including  $\mathcal{A}$ .

## Updating exchangeable models

The subject observes that  $\omega_1, \dots, \omega_n$  are exchangeable. This means that for any gamble  $f$  and any permutation  $\pi$ , he holds exchanging  $f$  by  $f \circ \pi$  weakly desirable, because he is indifferent between them. The negative natural extension of each such exchange gamble is

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If  $\mathcal{D}_w$  consists of weakly desirable gambles, then so does its conical hull  $\mathcal{D}_w(\mathcal{A})$ .

A subset  $\mathcal{A}$  of all exchange gambles on  $\mathbb{R}^n$  is called exchangeable if  $\mathcal{D}_w \subseteq \mathcal{D}_w(\mathcal{A})$  or equivalently if

$$\mathcal{D}_w \subseteq \mathcal{D}_w(\mathcal{A})$$

If  $\mathcal{A}$  is coherent and exchangeable then it is also permutation for  $\mathcal{A} \cap \mathcal{D}_w$  and all  $\pi \in \mathcal{D}_w$  holds that  $\mathcal{A} \cap \mathcal{D}_w = \mathcal{A} \cap \mathcal{D}_w \circ \pi$ .

## Exchangeable previsions

A lower prevision  $\underline{E}$  on  $\mathbb{R}^n$  is exchangeable if there is some non-possibility coherent set of desirable gambles  $\mathcal{A}$  such that  $\underline{E}(f) = \overline{E}(f)$ .

## Moving between sequence gambles and count gambles

The set of permutation invariant sequence gambles is

$$\mathcal{D}_w(\mathcal{A}) := \{f \in \mathbb{R}^n \mid f = f \circ \pi, \pi \in \mathcal{D}_w\}$$

The projection of a sequence gamble  $f$  onto a permutation invariant gamble is

$$\text{proj}(f) := \sum_{\pi \in \mathcal{D}_w} f \circ \pi$$

whose value on an individual  $\omega$  is given by

$$\text{proj}(f)(\omega) := \sum_{\pi \in \mathcal{D}_w} f(\omega \circ \pi)$$

The permutation invariant sequence gamble is a one-to-one correspondence with the count gamble

$$\mathcal{D}_w(\mathcal{A}) := \{p \in \mathbb{R}^n\}$$

## Representation

A set of desirable gambles  $\mathcal{D}$  on  $\mathbb{R}^n$  is coherent and exchangeable if there is some coherent set  $\mathcal{A}$  of desirable gambles on  $\mathbb{R}^n$  is permutation-invariant such that

$$\mathcal{D} = \text{conv}(\mathcal{A}) \cap \mathcal{D}_w$$

and in that case this  $\mathcal{A}$  is uniquely determined by

$$\mathcal{A} := \{f \in \mathbb{R}^n \mid f \in \mathcal{D}, f \circ \pi \in \mathcal{D}, \pi \in \mathcal{D}_w\}$$

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$$\mathcal{A} := \{f \in \mathbb{R}^n \mid f \in \mathcal{D}, f \circ \pi \in \mathcal{D}, \pi \in \mathcal{D}_w\}$$

## Representing updated models

The subject observes the values  $\omega_1, \dots, \omega_n$  of the count vector  $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$  if the first  $n$  variables  $\omega_1, \dots, \omega_n$ .

If  $\mathcal{D}$  is a coherent and exchangeable set of desirable gambles on  $\mathbb{R}^n$ , then the representation of the lower bounds of exchangeable updated models to view is

$$\mathcal{D}_w(\mathcal{A}) := \{f \in \mathbb{R}^n \mid f \in \mathcal{D}, f \circ \pi \in \mathcal{D}, \pi \in \mathcal{D}_w\}$$

This representation is not an updated model of the representation  $\mathcal{D} = \text{conv}(\mathcal{A}) \cap \mathcal{D}_w$  of  $\mathcal{D}$ . They are however related by

$$\mathcal{D}_w(\mathcal{A}) := \text{conv}(\mathcal{A}) \cap \mathcal{D}_w$$

when we use the identical function, defined for every count vector  $\omega \in \mathbb{R}^n$  by

$$\text{proj}(f)(\omega) := \sum_{\pi \in \mathcal{D}_w} f(\omega \circ \pi)$$

which is zero when  $\omega = 0$ .

## Exchangeable previsions & representation

A lower prevision  $\underline{E}$  on  $\mathbb{R}^n$  is coherent and exchangeable if there is some coherent lower prevision  $\underline{E}$  on  $\mathbb{R}^n$  is permutation-invariant such that  $\underline{E}(f) = \overline{E}(f)$ . In that case  $\underline{E}$  is uniquely determined by  $\underline{E}(f) = \overline{E}(f)$ .

Representation

- ▶ representation theorem
- ▶ exchangeable natural extension & representation
- ▶ representing updated exchangeable models
- ▶ relationship with representing previsions



# Exchangeability for sets of desirable gambles

Desirability

## General context: experiments & gambles

A from possibility space  $\mathcal{X}$  of outcomes of some experiment.  
A subject sets a collection about the experiment's outcome.  
Satisfies  $f \in \mathcal{F} \Rightarrow \mathbb{1}_X \in \mathcal{F}$ , interpreted as gambler's receipt:  
 $f(x)$  when the experiment's outcome is  $x$ .



A gambler  $g$  is admitted to bet subject if he satisfies the following characteristics:  
(1) The actual outcome  $x$  is determined, and  
(2) The subject's receipt is changed by  $f(x)$ .  
The new gambler  $g'$  is not desirable.

## Coherent sets of desirable gambles

A subject's set of desirable gambles  $\mathcal{D} \subseteq \mathcal{F}(\mathcal{X})$  models his beliefs about the experiment's outcome.

- The set of desirable gambles  $\mathcal{D}$  is coherent if it satisfies the following rationality requirements:
- $\mathbb{1}_X \in \mathcal{D}$  and  $\mathcal{D} \neq \emptyset$ .
  - $\mathcal{D}$  is a cone:  $f \in \mathcal{D} \Rightarrow \lambda f \in \mathcal{D}$ ,  $\lambda > 0$ .
  - $\mathcal{D}$  is closed under addition of permutation gambles.
  - $\mathcal{D}$  is closed under pointwise multiplication.
  - $\mathcal{D}$  is closed under pointwise multiplication.
  - $\mathcal{D}$  is closed under pointwise multiplication.



## Sets of weakly desirable gambles

The subject considers a gamble  $f$  in  $\mathcal{F}(\mathcal{X})$  weakly desirable if he holds any desirable gamble  $g$  & another desirable gamble  $h$  observed, so  $f \in \mathcal{D}$  if then  $f + g \in \mathcal{D}$ .

The subject's set of weakly desirable gambles is  $\mathcal{D}_w = \{f \in \mathcal{F}(\mathcal{X}) : f + g \in \mathcal{D}\}$ .  
The set of weakly desirable gambles  $\mathcal{D}_w$  corresponding to a coherent set of desirable gambles  $\mathcal{D}$  satisfies the following properties:  
(1)  $\mathbb{1}_X, h \in \mathcal{D}_w$  ( $\mathbb{1}_X, h \in \mathcal{D}$ ).  
(2)  $\mathcal{D}_w$  is a cone:  $f \in \mathcal{D}_w \Rightarrow \lambda f \in \mathcal{D}_w$ ,  $\lambda > 0$ .  
(3)  $\mathcal{D}_w$  is closed under addition of permutation gambles.  
(4)  $\mathcal{D}_w$  is closed under pointwise multiplication.  
(5)  $\mathcal{D}_w$  is closed under pointwise multiplication.



## Assessments & their natural extension

An assessment can consist of a set  $\mathcal{A} \subseteq \mathcal{F}(\mathcal{X})$  considered desirable by the subject.

The assessment of a gamble  $g$  under non-desirability of the intersection of  $\text{conv}(\mathcal{A})$  &  $\mathcal{D}_w$  is empty.  
The natural extension of  $\mathcal{A}$  is  $\mathcal{D}_w(\mathcal{A}) := \text{conv}(\mathcal{A}) \cap \mathcal{D}_w(\mathcal{A})$ .



If  $\mathcal{A}$  is desirably non-desirable, then  $\mathcal{D}_w(\mathcal{A})$  is the smallest coherent set of desirable gambles including  $\mathcal{A}$ .

## Updating sets of desirable gambles

The subject observes, or considers the possibility of observing, an event  $B \subseteq \mathcal{X}$ .  
Contingent on observing  $B$ , the subject models his beliefs using a coherent set of desirable gambles. The subject's  $\mathcal{D}(B)$  given by  $\mathcal{D}(B) = \{g \in \mathcal{F}(B) : g \in \mathcal{D}\}$ .

If  $\mathcal{A}$  is a coherent set of desirable gambles on  $\mathcal{X}$ , then  $\mathcal{D}_w(\mathcal{A})$  is a coherent set of desirable gambles on  $\mathcal{X}$ .

## Coherent previsions & desirability

The lower prevision of a gamble  $f$  established to a set of desirable gambles  $\mathcal{D}$  is  $\underline{P}(f) := \sup\{c \in \mathbb{R} : f - c \in \mathcal{D}\}$ .

Its conjugate upper prevision  $\overline{P}(f)$  is  $\overline{P}(f) := \inf\{c \in \mathbb{R} : c - f \in \mathcal{D}\}$ .

A lower prevision  $\underline{P}$  is coherent if there exists some coherent set of desirable gambles  $\mathcal{D}$  such that  $\underline{P}(f) = \underline{P}(f)$ .

Coherent lower previsions are less expressive uncertainty models than coherent sets of desirable gambles.



Exchangeability

## Specific context: finite sequences

The experiment consists of the observation of the value of a random  $X_1, \dots, X_n$  of random variables to which  $\mathcal{X}$  is the finite set of possible values. So the possibility space is  $\mathcal{X} = \{(\omega_1, \dots, \omega_n) : \omega_i \in \mathcal{X}_i, \omega_i \in \mathcal{X}_i, \omega_i \in \mathcal{X}_i\}$ .

$\mathcal{D}_w$  is the set of all permutations  $\sigma$  of the index set  $\{1, \dots, n\}$ .

The associated permutation of  $\mathcal{D}$  is defined by  $\mathcal{D}_w = \text{conv}(\mathcal{D})$ . It is called a permutation of  $\mathcal{D}$  if  $\mathcal{D} = \mathcal{D}_w$ .

With every sequence of observations corresponds a joint vector in  $\mathbb{R}^n = \{x \in \mathbb{R}^n : x_i \in \mathcal{X}_i, \omega_i \in \mathcal{X}_i\}$ .

The corresponding  $\mathcal{D}_w = \mathcal{D}_w$  is a cone:  $\mathcal{D}_w = \text{conv}(\mathcal{D})$ .

Permitted gambles have the same joint vector: a permutation coherent gamble is  $\mathcal{D}_w = \text{conv}(\mathcal{D})$ .

$$\mathcal{D}_w = \{x \in \mathbb{R}^n : P(x) = a\}$$

$$\mathcal{D}_w = \{x \in \mathbb{R}^n : P(x) = a\}$$

## Exchangeability

A subject assesses that  $X_1, \dots, X_n$  are exchangeable. This means that for any gamble  $g$  and any permutation  $\sigma$  he holds exchanging  $\sigma f$  for  $f$  weakly desirable, because he is indifferent between them. The negative mean response of all such exchange gambles is  $\mathcal{D}_w = \{f - \sigma f : f \in \mathcal{D}_w\}$ .

A  $\mathcal{D}_w$  consists of weakly desirable gambles, then so does its conical hull  $\mathcal{D}_w = \text{conv}(\mathcal{D}_w)$ .

A subset  $\mathcal{A}$  of all exchange gambles on  $\mathcal{X}^n$  is called exchangeable if  $\mathcal{D}_w = \mathcal{D}_w$  or equivalently  $\mathcal{D}_w = \text{conv}(\mathcal{A})$ .

If  $\mathcal{A}$  is coherent and exchangeable then it is also permutation for all  $f \in \mathcal{A}$  and all  $\sigma$  in  $\mathcal{S}_n$  holds that  $\sigma f \in \mathcal{A}$ .

## Exchangeable natural extension

The assessment of a gamble  $g$  under non-desirability of the intersection of  $\text{conv}(\mathcal{A})$  &  $\mathcal{D}_w$  is empty.

The exchangeable natural extension of  $\mathcal{A}$  is  $\mathcal{D}_w(\mathcal{A}) := \text{conv}(\mathcal{A}) \cap \mathcal{D}_w(\mathcal{A})$ .

If  $\mathcal{A}$  is desirably non-desirable under exchangeability, then  $\mathcal{D}_w(\mathcal{A})$  is the smallest exchangeable coherent set of desirable gambles including  $\mathcal{A}$ .

## Updating exchangeable models

The subject observes the values  $x = (x_1, \dots, x_n)$  on the joint vector  $\omega \in \mathcal{X}$  or the first  $n$  variables  $X_1, \dots, X_n$  the mean observing the event  $\{x \in \mathcal{X} : \omega_1 = x\} = \mathcal{B}$ . She is interested in observations about the remaining  $n - k$  variables.

Contingent on observing  $x$  on  $\mathcal{B}$ , the subject models his beliefs using updated sets of desirable gambles. The subsets of  $\mathcal{D}(B)$  that are  $\mathcal{D}(B) = \{g \in \mathcal{F}(B) : g \in \mathcal{D}\}$  is available.

Contingent on observing  $x$  on  $\mathcal{B}$ , the subject models his beliefs using updated sets of desirable gambles. The subsets of  $\mathcal{D}(B)$  that are  $\mathcal{D}(B) = \{g \in \mathcal{F}(B) : g \in \mathcal{D}\}$  is available.

If  $\mathcal{A}$  is a coherent and exchangeable set of desirable gambles on  $\mathcal{X}^n$ , then  $\mathcal{D}_w(\mathcal{A})$  is a coherent and exchangeable set of desirable gambles on  $\mathcal{X}^n$ .

Under exchangeability, joint vectors are sufficient statistics:  $\mathcal{D}_w(\mathcal{A}) = \text{conv}(\mathcal{A}) \cap \mathcal{D}_w(\mathcal{A})$ .

## Exchangeable previsions

A lower prevision  $\underline{P}$  on  $\mathcal{F}(\mathcal{X}^n)$  is exchangeable if there is some exchangeable coherent set of desirable gambles  $\mathcal{D}$  such that  $\underline{P}(f) = \underline{P}(f)$ .

## Moving between sequence gambles and count gambles



The set of permutation invariant sequence gambles is  $\mathcal{D}_w = \{f \in \mathcal{F}(\mathcal{X}^n) : f = \sigma f \text{ for all } \sigma \in \mathcal{S}_n\}$ .

The projection of a sequence gamble  $f$  onto a permutation invariant sequence gamble is  $\mathcal{D}_w = \{f \in \mathcal{F}(\mathcal{X}^n) : f = \sigma f \text{ for all } \sigma \in \mathcal{S}_n\}$ .

The count gamble corresponding to the sequence gamble  $f$  is  $\mathcal{D}_w = \{f \in \mathcal{F}(\mathcal{X}^n) : f = \sigma f \text{ for all } \sigma \in \mathcal{S}_n\}$ .

The permutation invariant sequence gamble  $f$  is a one-to-one correspondence with the count gamble  $g$ :  $\mathcal{D}_w = \{f \in \mathcal{F}(\mathcal{X}^n) : f = \sigma f \text{ for all } \sigma \in \mathcal{S}_n\}$ .

$\mathcal{D}_w = \{f \in \mathcal{F}(\mathcal{X}^n) : f = \sigma f \text{ for all } \sigma \in \mathcal{S}_n\}$ .

## Representation

A set of desirable gambles  $\mathcal{D}$  on  $\mathcal{X}^n$  is coherent and exchangeable if there is some coherent set  $\mathcal{A}$  of desirable gambles on  $\mathcal{X}^n$  - its representation - such that  $\mathcal{D} = \text{conv}(\mathcal{A}) \cap \mathcal{D}_w$ .

and in that case this  $\mathcal{A}$  is uniquely determined by  $\mathcal{A} = \{x \in \mathcal{X}^n : P(x) = a\}$ .

## Exchangeable natural extension & representation

The assessment of a gamble  $g$  under non-desirability of the intersection of  $\text{conv}(\mathcal{A})$  &  $\mathcal{D}_w$  is empty.

A count result  $\mathcal{D}_w = \{f \in \mathcal{F}(\mathcal{X}^n) : f = \sigma f \text{ for all } \sigma \in \mathcal{S}_n\}$ .

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## Representing updated models

The subject observes the values  $x = (x_1, \dots, x_n)$  on the joint vector  $\omega \in \mathcal{X}$  or the first  $n$  variables  $X_1, \dots, X_n$ .

If  $\mathcal{A}$  is a coherent and exchangeable set of desirable gambles on  $\mathcal{X}^n$ , then the representation of the lower - because of exchangeability - identical updated models to use is  $\mathcal{D}_w = \text{conv}(\mathcal{A}) \cap \mathcal{D}_w$ .

This representation is not an updated model of the representation  $\mathcal{D}_w = \text{conv}(\mathcal{A}) \cap \mathcal{D}_w$  of  $\mathcal{A}$ . They are however related by  $\mathcal{D}_w = \text{conv}(\mathcal{A}) \cap \mathcal{D}_w$ .

when we use the dual functional, defined for every count vector  $x \in \mathcal{X}^n$  by  $\mathcal{D}_w = \{f \in \mathcal{F}(\mathcal{X}^n) : f = \sigma f \text{ for all } \sigma \in \mathcal{S}_n\}$ .

which is zero when  $j \neq n$ .

$\mathcal{D}_w = \{f \in \mathcal{F}(\mathcal{X}^n) : f = \sigma f \text{ for all } \sigma \in \mathcal{S}_n\}$ .

## Exchangeable previsions & representation

A lower prevision  $\underline{P}$  on  $\mathcal{F}(\mathcal{X}^n)$  is coherent and exchangeable if there is some coherent set of desirable gambles  $\mathcal{D}$  on  $\mathcal{X}^n$  - its representation - such that  $\underline{P}(f) = \underline{P}(f)$ . In that case  $\mathcal{A}$  is uniquely determined by  $\mathcal{A} = \{x \in \mathcal{X}^n : P(x) = a\}$ .

Representation

See you at the poster!