Exchangeability for sets of desirable gambles

Gert de Cooman & Erik Quaeghebeur
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Exchangeable lower previsions

1. Introduction

This paper deals with belief models for both finite and countable samples of exchangeable random variables taking a finite number of values. When such sequences of random variables are assumed to be exchangeable, the more or less means that the specific order in which they are observed is deemed irrelevant. This means that a priori beliefs of, say, a gambler. He is only trying to predict, or make inferences about whether the next ball will be 'white' or 'red', whereas the specific order in which these predictions are made does not matter for his predictions. So we let our subject consider a finite and non-empty set \( \Omega \) of possible outcomes, or 'types', and we then say that the belief model is exchangeable if it is conditionally independent and identically distributed (C.I.I.D.).

We also discuss a few examples to show the relevance of our findings and we then say that the belief model is exchangeable if it is conditionally independent and identically distributed (C.I.I.D.).

One of the reasons why exchangeability is deemed important, especially by Bayesians, is because it is a fundamental property of the traditional Bayesian belief models, but where this fundamental difference can be captured. This leads to two notions of symmetry for such traditional Bayesian belief models, but where this fundamental difference can be captured.

This echoes Walley's [1991, Section 9.5.6, p. 466] view that 'symmetry of evidence' reflects the above-mentioned exchangeability assessment.

The system \( \Pi \) of predictive lower previsions:

\[
\Pi = \{ \lambda \mid \lambda : \mathcal{P}(\Omega) \rightarrow [0,1] \text{ is a lower probability function} \}
\]

is a finite and non-empty set

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\]

in the set of all gambles

\[
\mathcal{P}(\Omega) = \{ \{ A_i \} \mid A_i \in \Omega, 1 \leq i \leq n \}
\]

and \( n \) is a finite and non-empty set

\[
\mathcal{P}(\Omega) = \{ \{ A_i \} \mid A_i \in \Omega, 1 \leq i \leq n \}
\]

of 'equivalent' events). Therein was proven the now famous representation theorem, which is often considered the cornerstone of the imprecise probability approach. However, even the most basic and straightforward properties of exchangeability can be a source of new insight, and that our system can be used to capture the reasoning of exchangeable systems (Heath and Allsop 1921, 1922).

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One of the reasons why exchangeability is deemed important, especially by Bayesians, is because it is a fundamental property of the traditional Bayesian belief models, but where this fundamental difference can be captured.
- (in)finite sequences of finite-valued random variables
- exchangeability assessment & sequence order permutations
- sample sequences, count vectors & frequency vectors
- representation theorems
- exchangeable natural extension
Immediate prediction under exchangeability and representation insensitivity

Gert de Cooman & Enrique Miranda

1. The setting

We consider a subject's beliefs about a sequence of independent random variables, which is assumed to be exchangeable (see De Groot, Section 9.5.6, p. 466) for a critical discussion of this notion of exchangeability.

Immediate prediction involves the prediction of future observations based on past observations. In particular, it considers the probability mass function of the next observation, given past observations. This leads to two notions of symmetry for such predictions. The first notion of symmetry is that of representation invariance, which essentially asserts that the specific order in which variables are observed is deemed irrelevant.

2. Requirements & Assumptions

One of the reasons why exchangeability is deemed important, especially by Bayesians, is that it ensures the invariance of the predictive lower previsions with respect to the specific order in which variables are observed. This more or less means that the specific order in which they are observed is deemed irrelevant.

Immediate prediction can be captured by a predictive probability mass function. They in turn correspond to a predictive lower prevision. We study when an exchangeable predictive system is related to the imprecise Dirichlet-Multinomial model.

The imprecise Dirichlet-Multinomial model can be used to represent uncertainty about the distribution of a finite sequence of observations. It is a generalization of the Dirichlet-Multinomial model, which assumes exchangeability.

3. Representation insensitivity in immediate prediction

Inference: Essays in Honor of Henry E. Kyburg, Jr.

Enrique Miranda: representation insensitivity , Imprecise Dirichlet-Multinomial model: symmetric exchangeability

We also discuss a few examples to show the relevance of our findings.

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Representation insensitivity in immediate prediction under exchangeability

Gert de Cooman & Enrique Miranda

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We also discuss a few examples to show the relevance of our findings.
Immediate prediction under exchangeability …

- families of exchangeable lower previsions
- coherent updated exchangeable lower previsions
- count vectors as sufficient statistics
Immediate prediction under exchangeability ...
Exchangeability for sets of desirable gambles

**Desirability**
- A subject investigates the desirability of a set of gambles. The set of desirable gambles is denoted as \( \Omega \).
- A gamble \( g \) is desirable if there is no other gamble \( h \) that is more desirable.

**Coherent sets of desirable gambles**
- A coherent set is a set of desirable gambles that is consistent with the axioms of rationality. It is the smallest consistent set of desirable gambles.

**Sets of weakly desirable gambles**
- A weakly coherent set is a set of desirable gambles that is consistent with the axioms of rationality, but not necessarily coherent.

**Updating sets of desirable gambles**
- An updated coherent set is a coherent set of desirable gambles that is consistent with the axioms of rationality.

**Exchangeability**
- The concept of exchangeability is introduced, which is the property of a set of gambles that the desirability of the gambles remains unchanged under permutations.

**Exchangeable models**
- An exchangeable model is a model where the desirability of the gambles remains unchanged under permutations.

**Exchangeable extensions**
- An exchangeable extension is the extension of an exchangeable model that is consistent with the axioms of rationality.

**Exchangeable representations**
- An exchangeable representation is the representation of an exchangeable model that is consistent with the axioms of rationality.

**Relationship with previsions**
- The relationship between exchangeability and previsions is discussed, highlighting the coherence and exchangeability of the sets of desirable gambles.

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Exchangeability for sets of desirable gambles

Sets of weakly desirable gambles

The subject considers a gamble \( f \) in \( \Omega \). He finds exchanging \( f \) for any desirable gamble avoids non-positivity under exchangeability, then \( f \) is weakly desirable.

Updating sets of desirable gambles

If \( f \) is a coherent and exchangeable set of desirable gambles on \( X \), then \( f \) must be a subset of \( \mathbb{D}_\mathbb{E}(X) \).

Coherent provisions & desirability

The lower provision of a gamble \( f \) associated to a set of desirable gambles \( \mathbb{D} \) is:

\[
C(f, x) = \min_{\mathbb{D} \subseteq \mathbb{D}_\mathbb{E}(X)} \{ c \in \mathbb{R} : \mathbb{D}_{\mathbb{D}} \vdash c \}
\]

Assessments & their natural extension

An assessment \( \mathbb{G} \) consists of weakly desirable gambles, then so does its coherently extended set.

Coherent sets of desirable gambles

A subject who is uncertain about the experiment's outcome.

Exchangeability

The associated permutation of \( X \) is:

\[
\pi(x) = \arg\max_{y \in X} P(y | x)
\]

Exchangeable natural extension

The exchangeable lower provision of \( f \) is:

\[
\tilde{C}(f, x) = \min_{\mathbb{D} \subseteq \mathbb{D}_\mathbb{E}(X)} \{ c \in \mathbb{R} : \mathbb{D} \vdash c \}
\]

Exchangeability assessment & weakly desirable gambles

Exchangeable natural extension

Updating exchangeable models

Exchangeable provisions

A lower provision \( \mathbb{G} \) is exchangeable if its exchangeable lower provision is coherent.

Updating exchangeable models

The subject observes the values \( x_1, x_2, \ldots, x_n \) of the count vector \( X = (X_1, X_2, \ldots, X_n) \). He receives an updated set of desirable gambles, defined by:

\[
\mathbb{D}_{\mathbb{F}} = \{ g : g(x) = \tilde{C}(g, x) \}
\]

Exchangeable natural extension

If \( f \) is coherent and exchangeable, then its exchangeable set is:

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Updating exchangeable models

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Exchangeability for sets of desirable gambles

Sets of weakly desirable gambles

The subject observes a gambles \( f \) (or \( f' \)), weakly desirable if by adding any desirable gamble to it, another desirable gamble is obtained, so \( f' \cap f' \neq \emptyset \) then \( f' \cap f \neq \emptyset \).

The set of weakly desirable gambles \( D \) corresponding to a coherent set of desirable gambles \( S \) satisfies the following properties:

1. \( \emptyset \in D \)
2. If \( f, g \in D \) then \( f \cup g \in D \)
3. \( f \in D \) if and only if \( -f \in D \)

Updating sets of desirable gambles

If \( f' \) is composed of desirable gambles on \( \Omega \), then \( f' \) is a coherent set of desirable gambles on \( \Omega \).

Coherent previsions & desirability

The lower prevision of a gambles \( \hat{\omega} \) associated to a set of desirable gambles \( \omega \)

\[\hat{\omega} = \min_{\omega \in \omega} \omega \]

The upper prevision for \( \hat{\omega} \) is

\[\check{\omega} = \max_{\omega \in \omega} \omega \]

Exchangeability

If \( f \subseteq g \) then \( \check{f} \subseteq \check{g} \) and \( \hat{f} \subseteq \hat{g} \).

Updating exchangeable models

The subject observes the values \( x = \langle x_1, \ldots, x_n \rangle \) or the count vector \( n = \langle n_1, \ldots, n_k \rangle \) of the outcomes. The count vector \( n \) is interpreted as uncertain rewards:

\[\sum_{i=1}^{n} x_i = n \cdot \hat{\omega} \]

Exchangeable natural extension & representation

A lower prevision \( P \) is exchangeable if there is some exchangeable coherent set of desirable gambles \( \omega \).

Exchangeable representations

A lower prevision \( P \) is exchangeable if there is some exchangeable coherent set of desirable gambles \( \omega \).
Exchangeability for sets of desirable gambles

General context: experiments & gambles
A subject who is uncertain about the experiment’s outcome.
A gamble \( \Omega = (X, G, \text{prob}) \) is uncertain exchangeable if the probability of all events is equal.
A subject’s set of desirable gambles is not exchangeable.

Sets of weakly desirable gambles
A subject’s set of desirable gambles is weakly desirable if adding any desirable gamble to it, another desirable gamble is obtained, so \( D \subseteq D \) for all \( D \subseteq D \).

Updating sets of desirable gambles
A subject observes, on the possibility of observing, an event \( D \).

Infinite exchangeable sequences?

Best definition of coherence?

Representation for infinite exchangeable sequences?

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Best definition of coherence?

Representation for infinite exchangeable sequences?

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## Exchangeability for sets of desirable gambles

### General context: experiments & gambles
A subject who is uncertain about the experiment's outcome. A gamble $g$ in the set of desirable gambles $\mathcal{D}$ is represented by an uncertain reward $R(g)$ when the experiment outcome is $\omega$.

### Sets of weakly desirable gambles
The subject considers a gamble $g$ in $\mathcal{D}$ weakly desirable if adding any desirable gamble $h$ to $g$, another gamble is obtained, so $g + h$ is in $\mathcal{D}$. The subject set of weakly desirable gambles is $\mathcal{W} = (g + h : g \in \mathcal{D}, h \in \mathcal{D})$.

### Updating sets of desirable gambles
The subject observes, or considers the possibility of observing, an event $\Omega$.

### Coherent sets of desirable gambles
A subject's set of desirable gambles $\mathcal{D}$ maximizes his beliefs, i.e., $\mathcal{D} = \mathcal{D}^*$. The set $\mathcal{D}^*$ is uniquely determined by $\mathcal{D}$.

### Coherent previsions & desirability
The lower prevision of a gamble $g$ associated to a set of desirable gambles $\mathcal{D}$ is $\mathcal{P}^*(g) = \sum_{\omega} P(\omega) g(\omega)$. The coherent lower prevision of $g$ is $\mathcal{P}^*(g) = \sum_{\omega} P(\omega) g(\omega)$.

### Exchangeability
The subject assesses the sets $\{\omega_1, \ldots, \omega_n\}$ as exchangeable if the set $\{\sigma(\omega_1), \ldots, \sigma(\omega_n)\}$ is exchangeable for every permutation $\sigma$ of the index set $\{1, \ldots, n\}$.

### Exchangeable previsions & representation
A lower prevision $\mathcal{P}(g)$ on $g$ is exchangeable if there is some exchangeable coherent set of desirable gambles $\mathcal{D}^*$ such that $\mathcal{P}(g) = \mathcal{P}^*(g)$.