The desirability of desirability

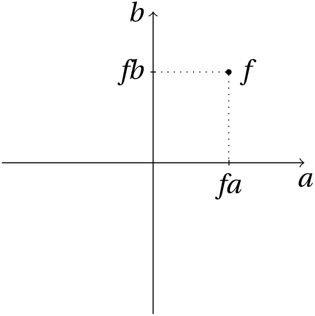
Erik Quaeghebeur

UNIVERSITEIT GENT

WAET-workshop of 19 December 2008

Gambles

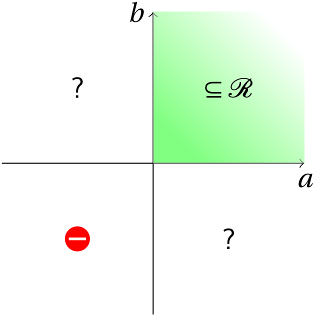
$$\Omega := \{a, b\}, \qquad f :\in \mathscr{L}_\Omega$$



Axioms of desirability: clear-cut cases

Accepting partial gains: $\begin{cases} \sup\{f\} > 0 \\ \inf\{f\} \geq 0 \end{cases} \Rightarrow f \in \mathscr{R}$

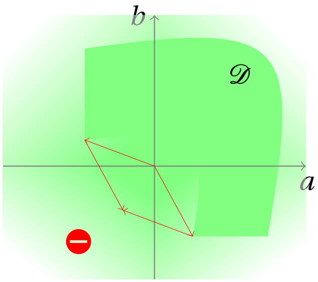
Avoiding partial loss: $\begin{cases} \inf\{f\} < 0 \\ \sup\{f\} \leq 0 \end{cases} \Rightarrow f \notin \mathscr{R}$



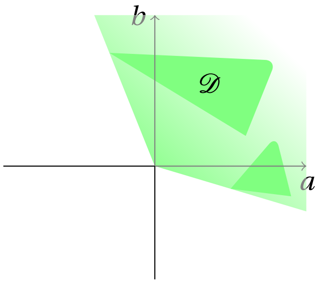
Axioms of desirability: how to extend

Positive scaling: $f \in \mathscr{R} \Rightarrow \lambda \cdot f \in \mathscr{R}$

Addition: $f, g \in \mathscr{R}^2 \Rightarrow f + g \in \mathscr{R}$



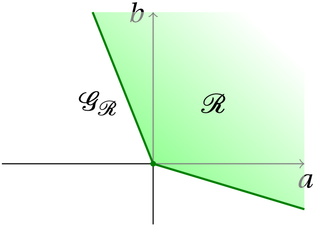
incurs sure loss



coherent extension

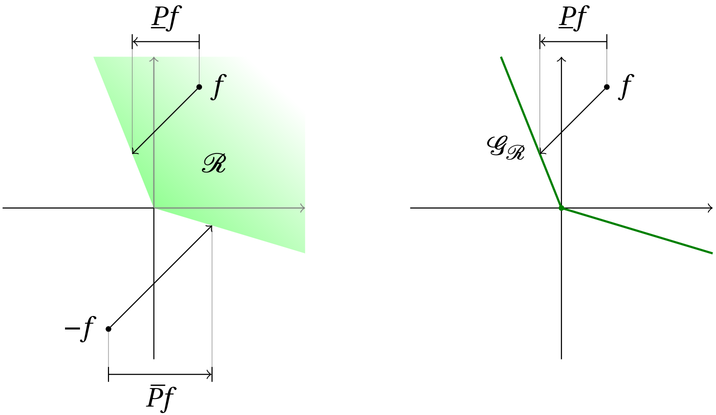
Marginally desirable gambles

$$\begin{aligned}\mathscr{G}_{\mathscr{D}} &:= \left\{f - \sup\{\alpha :\in \mathbb{R} \mid f - \alpha \in \mathscr{D}\} \mid f :\in \mathscr{D}\right\} \\ &= \left\{f - \sup\{\alpha :\in \mathbb{R}_{>0} \mid f - \alpha \in \mathscr{D}\} \mid f :\in \mathscr{D}\right\}\end{aligned}$$



From (marginally) desirable gambles to previsions

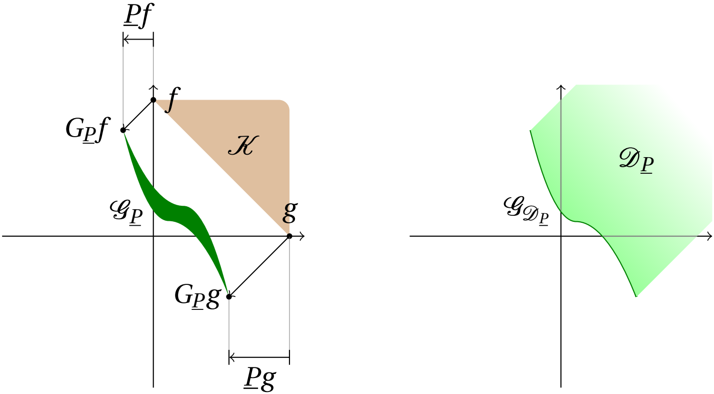
$$\underline{P}f := \sup\{\alpha :\in \mathbb{R} \mid f - \alpha \in \mathscr{R}\} = \sup\{\alpha :\in \mathbb{R} \mid \exists g :\in \mathscr{G}_{\mathscr{R}} ; f - \alpha \geq g\}$$
$$\overline{P}f := \inf\{\beta :\in \mathbb{R} \mid \beta - f \in \mathscr{R}\}$$



From previsions to (marginally) desirable gambles

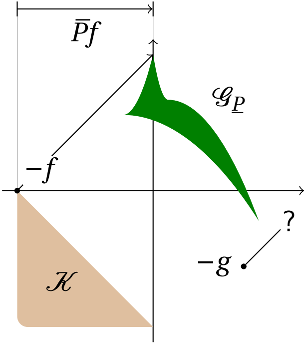
$$\mathscr{G}_{\underline{P}} := \{f - \underline{P}f \mid f :\in \mathscr{K}\} \qquad G_{\underline{P}}f = f - \underline{P}f$$

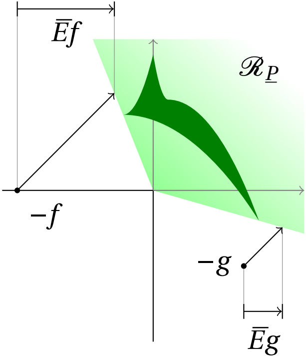
$$\mathscr{D}_{\underline{P}} := \mathscr{G}_{\underline{P}} + \mathbb{R}_{>0} = \{g + \alpha | g :\in \mathscr{G}_{\underline{P}}; \alpha :\in \mathbb{R}_{>0}\}$$



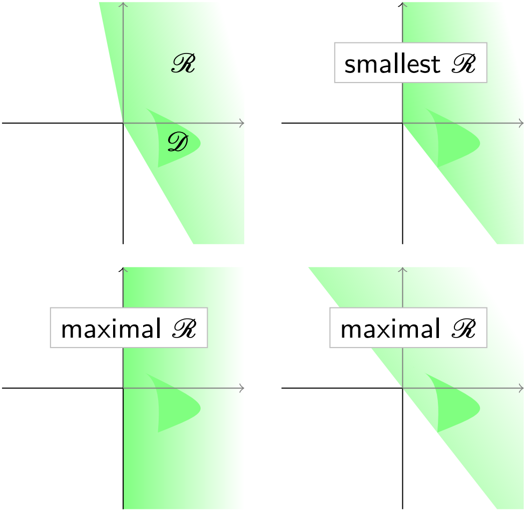
Natural extension of a prevision

$$\mathscr{R}_{\underline{P}} \coloneqq \mathscr{R}_{\mathscr{D}_{\underline{P}}}$$



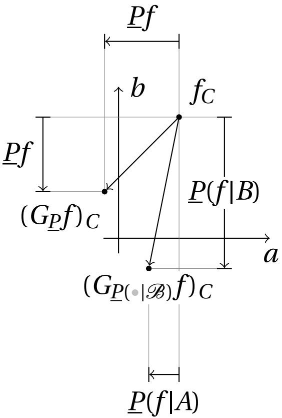


Least and maximally committal extensions



Marginal gambles of conditional previsions

$$G_{\underline{P}(\bullet|\mathscr{B})} f = f - \underline{P}(f|\mathscr{B})$$



$$\Omega := A \cup B$$

$$\mathscr{B} := \{A, B\}$$

$$C := \{a, b\}$$

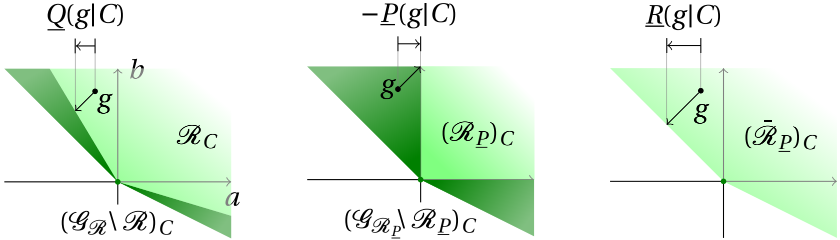
$$a \in A$$

$$b \in B$$

Updating on events of lower probability zero with natural and regular extension

$$\underline{P}C := \sup\{\alpha :\in \mathbb{R} \mid I^C - \alpha \in \mathscr{R}\} = 0$$

Any desirable gamble that has as its support an event of lower probability zero is *marginally* desirable.



$$\underline{Q}(g|C) := \sup\{\alpha :\in \mathbb{R} \mid g - \alpha \in \mathscr{R}_C\} > 0, \quad \mathscr{R}_C := \{f_C \mid f :\in \mathscr{R} \text{ such that } f \cdot I^C = f\}$$

$$\underline{P}(g|C) = \inf\{g\} \leq 0 < \underline{Q}(g|C)$$

$$\underline{R}(g|C) := \sup\{\alpha :\in \mathbb{R} \mid g - \alpha \in (\bar{\mathscr{R}}_{\underline{P}})_C\}, \quad \bar{\mathscr{R}}_{\underline{P}} := \mathscr{R}_{\underline{P}} \cup \{f :\in \mathscr{G}_{\mathscr{R}_{\underline{P}}} \mid \mathrm{mce}_{\underline{P}} f > 0\}$$

Defining exchangeability using (marginally) desirable gambles

$$\mathscr{H}\mathscr{X}^N := \operatorname{span}\{\pi f - f | f :\in \mathscr{L}_{\mathscr{X}^N}; \pi :\in \Pi_{1..N}\}$$

$$\boxed{\mathscr{H}\mathscr{X}^N + \mathscr{R}_{\neq(\mathscr{X}^N;0)} \subseteq \mathscr{R}}$$

The least committal exchangeable coherent set of desirable gambles is $\mathscr{H}\mathscr{X}^N + (\mathscr{L}_{\mathscr{X}^N})_{\geq 0 \wedge \neq (\mathscr{X}^N;0)}$.

$$\boxed{\begin{aligned} \operatorname{xch}\underline{P} \Leftrightarrow \mathscr{H}\mathscr{X}^N \subseteq \mathscr{G}_{\underline{P}} \Leftrightarrow \forall f :\in \mathscr{L}_{\mathscr{X}^N}; \\ \forall \pi :\in \Pi_{1..N}; \\ \underline{P}(\pi f - f) = 0 = \overline{P}(\pi f - f) \end{aligned}}$$

Representation theorems for exchangeable coherent sets of desirable gambles can be proven starting from this idea (recent results by Gert de Cooman).

The desirability of desirability

- A conceptually simple basis for imprecise-probability theory.
- Due to its geometric nature, it provides intuition in cases where this is more difficult to gain using the language of previsions.
- Deals elegantly with conditioning (slicing the cone through the conditioning event) and thus with the problem of events of lower probability zero.
- Therefore allows definitions of properties that hold for the entire uncertainty model (e.g., exchangeability), i.e., for updated previsions as well, even if the conditioning event has lower probability zero.

Questions

1. In desirability, conditioning is taking slices. It seems that taking slices of the cone are also important for marginalizing, or, more generally, the effect of applying a surjective map on the possibility space. Are there other types of slices than these two?
2. Lower previsions are less expressive than sets of desirable gambles. Lower previsions are less general than non-convex credal sets. It seems that the additional modeling power in each case addresses a different need. Can we formalize this difference? What would the uncertainty model look like that combines the additional modeling power of both?
3. Exchangeability can be elegantly defined using desirable gambles. It seems that this will also be the case for other types of symmetry assumptions. Are there other (types/classes of) concepts that can benefit from a redefinition using desirability?