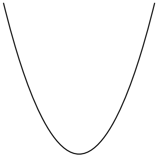
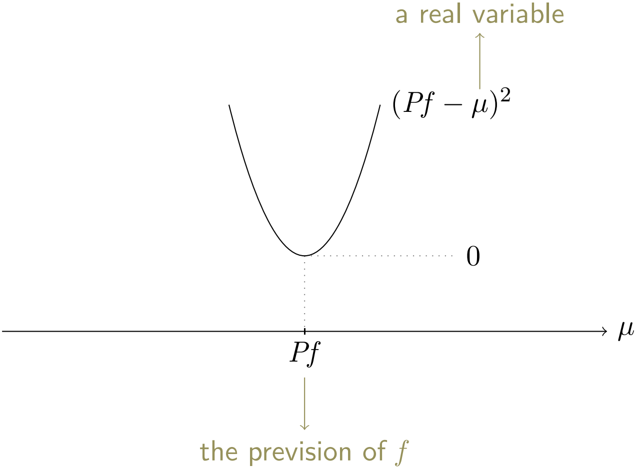
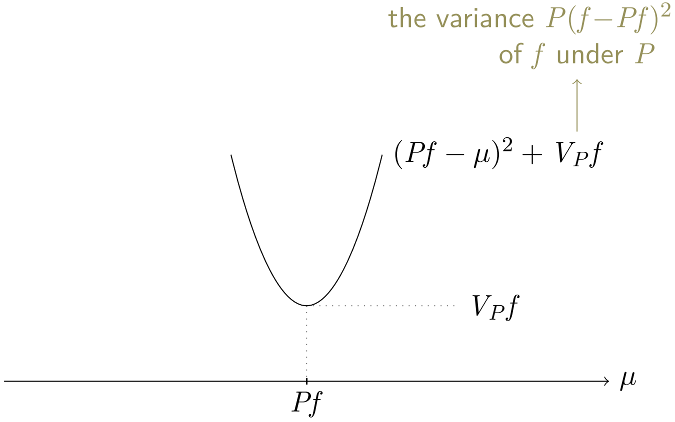


notation



P: a prevision (expectation operator)
f: a gamble (bounded real function)

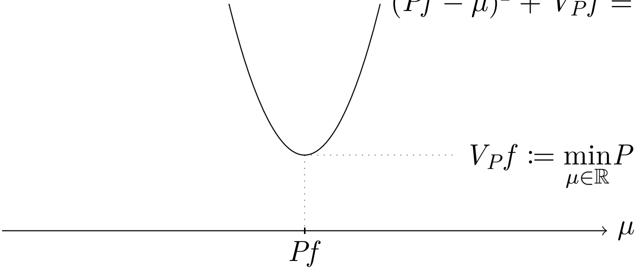
variance notation



variance

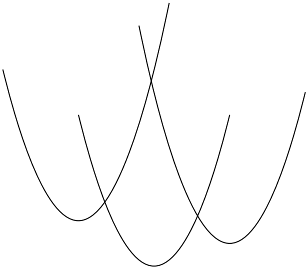
the variance $P(f-\mu+\mu-Pf)^2$
of f under P

$$(Pf-\mu)^2 + V_P f = P(f-\mu)^2$$



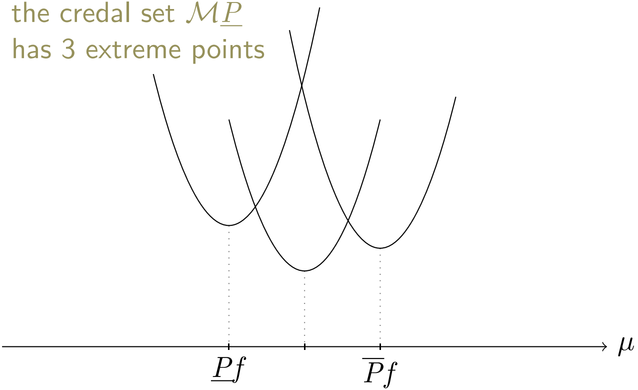
$$V_P f := \min_{\mu \in \mathbb{R}} P(f-\mu)^2$$

the variance of f under P
as an optimization problem
$(f-\mu)^2$: a gamble for every μ



notation

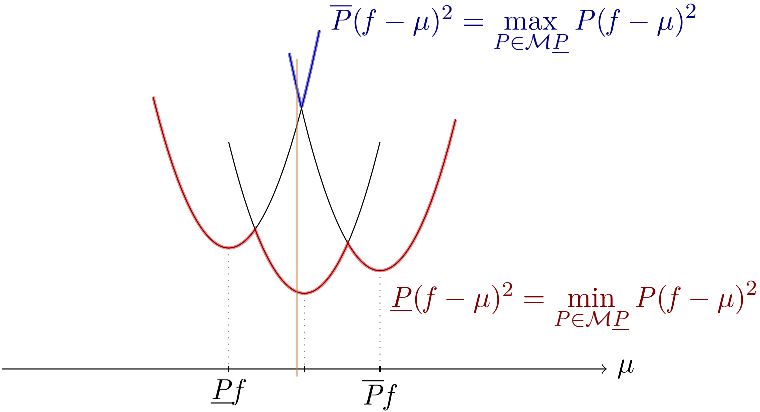
the credal set $\mathcal{M}\underline{P}$
has 3 extreme points



the lower prevision of f
$\underline{P}$: a lower prevision

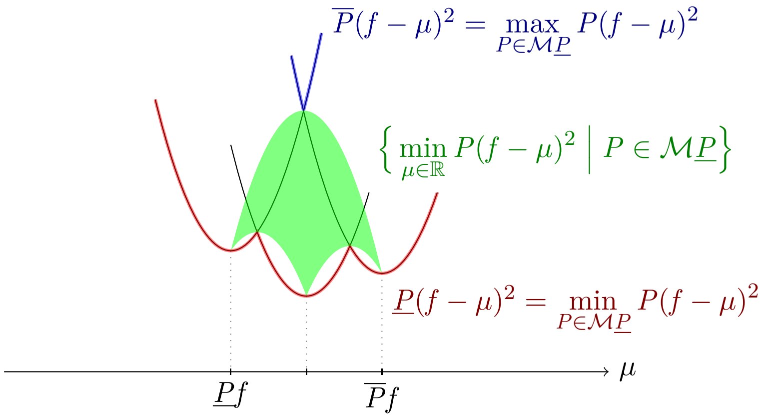
the upper prevision of f
$\overline{P}$: the conjugate upper prevision; $\overline{P}f = -\underline{P}(-f)$

envelopes

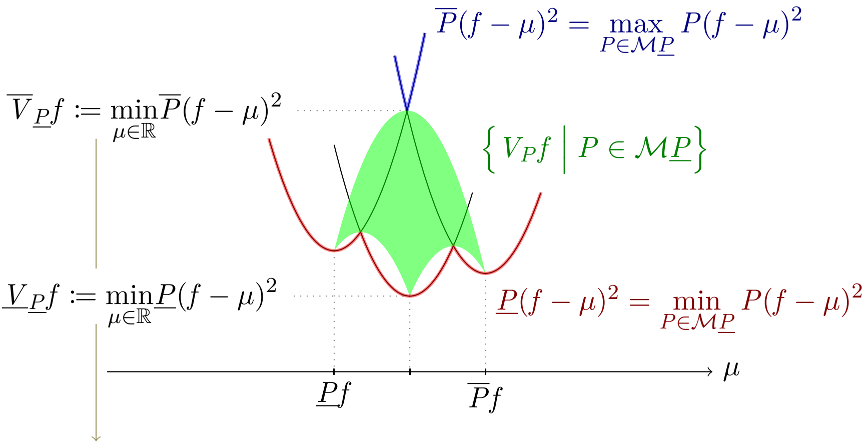


$\overline{P}(f-\mu)^2 = \max_{P \in \mathcal{M}\underline{P}} P(f-\mu)^2$
$\underline{P}(f-\mu)^2 = \min_{P \in \mathcal{M}\underline{P}} P(f-\mu)^2$
μ
$\underline{P}f$
$\overline{P}f$

envelopes and a set

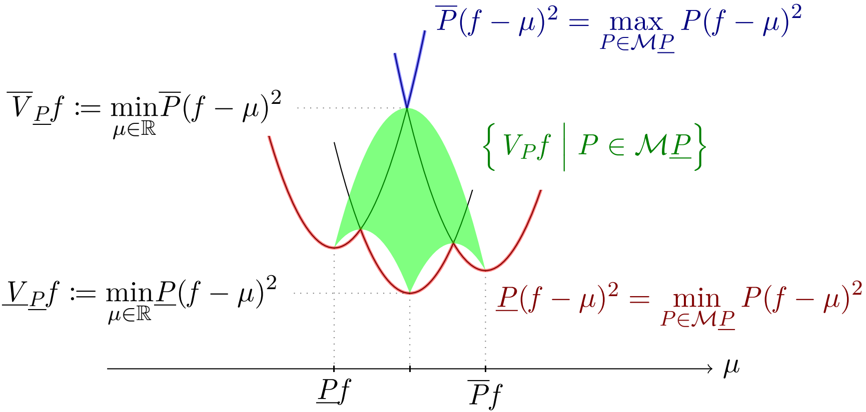


Lower & upper variance notation



the lower and upper variance of f under $\underline{P}$ as optimization problems

Lower & upper variance



Walley's variance envelope theorem:

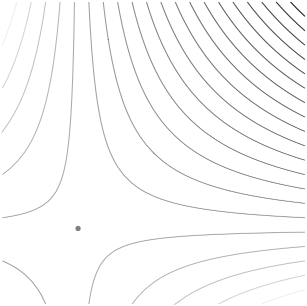
$$\underline{V}_{\underline{P}} f = \min_{P \in \mathcal{M}\underline{P}} V_P f \quad \text{and} \quad \overline{V}_{\underline{P}} f = \max_{P \in \mathcal{M}\underline{P}} V_P f.$$

Lower & upper **co**variance

Erik Quaeghebeur

UNIVERSITEIT GENT

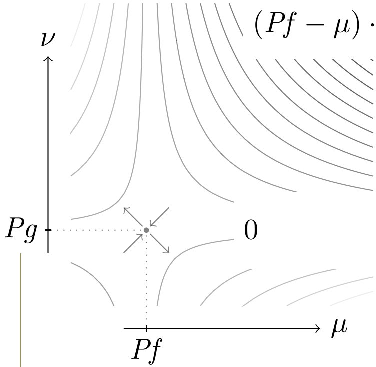
SMPS 2008



notation

a real variable

$(Pf - \mu) \cdot (Pg - \nu)$



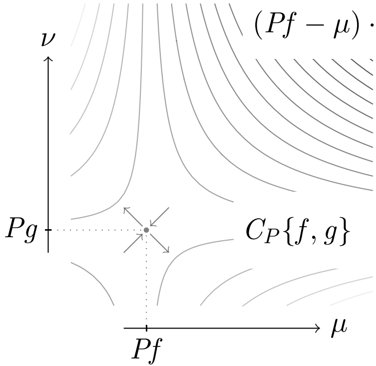
prevision of the gamble g

covariance notation

the covariance $P\big((f - Pf)\cdot(g - Pg)\big)$
of f and g under P

↑

$$(Pf - \mu)\cdot(Pg - \nu) + C_P\{f, g\}$$

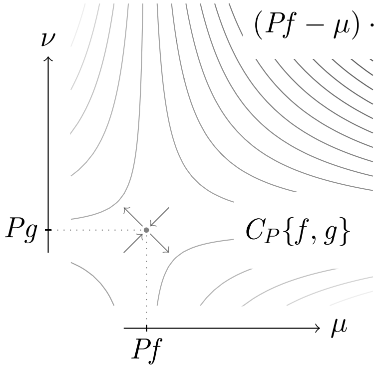


covariance

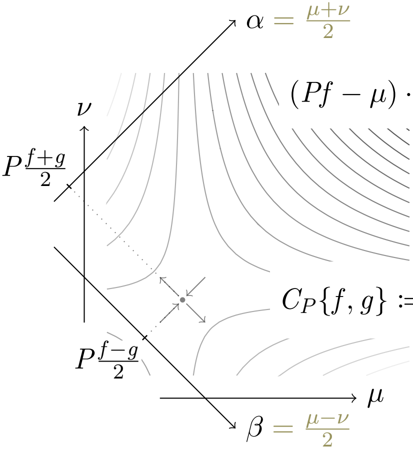
the covariance $P\big((f-\mu+\mu-Pf)\cdot(g-\nu+\nu-Pg)\big)$
of f and g under P

$\uparrow$

$$(Pf-\mu)\cdot(Pg-\nu)+C_P\{f,g\}$$
$$= P\big((f-\mu)\cdot(g-\nu)\big)$$



covariance

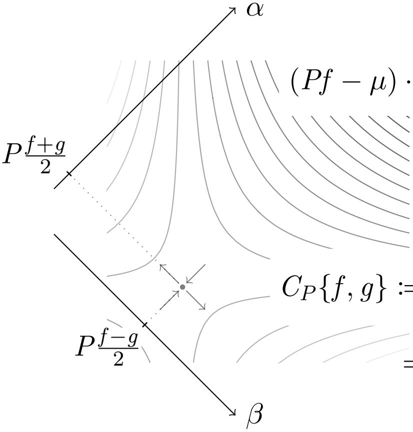


$$(Pf - \mu) \cdot (Pg - \nu) + C_P\{f, g\} = P\big((f - \mu) \cdot (g - \nu)\big)$$

$$C_P\{f, g\} := \min_{\alpha \in \mathbb{R}} \max_{\beta \in \mathbb{R}} P\big((\tfrac{f+g}{2} - \alpha)^2 - (\tfrac{f-g}{2} - \beta)^2\big)$$

the covariance of f and g under P
as an optimization problem
$(\frac{f+g}{2} - \alpha)^2 - (\frac{f-g}{2} - \beta)^2$:
a gamble for every α and β

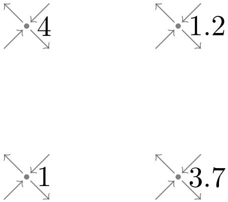
covariance

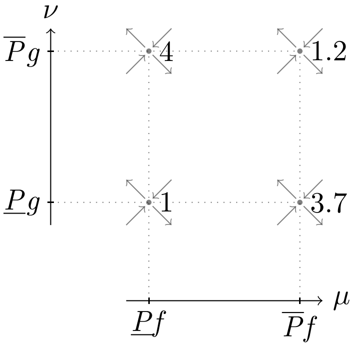


$$(Pf-\mu)\cdot(Pg-\nu)+C_P\{f,g\} = P\big((f-\mu)\cdot(g-\nu)\big)$$

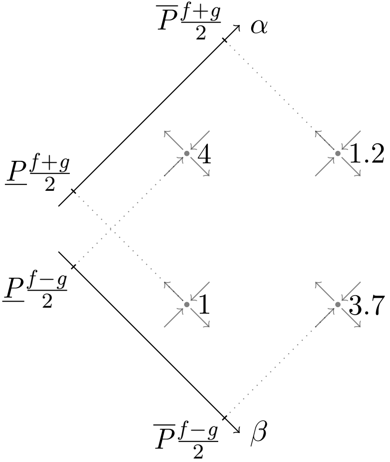
$$C_P\{f,g\} := \min_{\alpha\in\mathbb{R}}\max_{\beta\in\mathbb{R}} P\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big)$$

$$= V_P\frac{f+g}{2} - V_P\frac{f-g}{2}$$


4
1.2
1
3.7

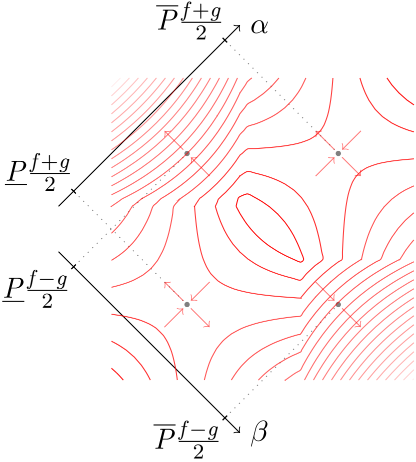


the credal set $\mathcal{M}\underline{P}$
has 4 extreme points



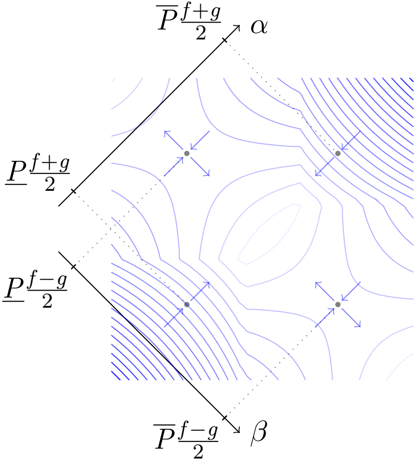
the credal set $\mathcal{M}\underline{P}$
has 4 extreme points

envelopes



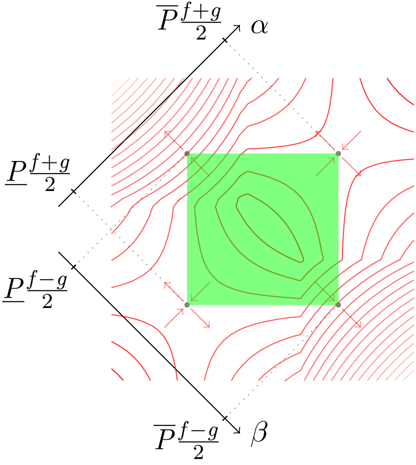
$$\underline{P}\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big) = \min_{P\in\mathcal{M}\underline{P}} P\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big)$$

envelopes



$$\overline{P}\left(\left(\tfrac{f+g}{2}-\alpha\right)^2-\left(\tfrac{f-g}{2}-\beta\right)^2\right)$$
$$=\max_{P\in\mathcal{M}\underline{P}} P\left(\left(\tfrac{f+g}{2}-\alpha\right)^2-\left(\tfrac{f-g}{2}-\beta\right)^2\right)$$

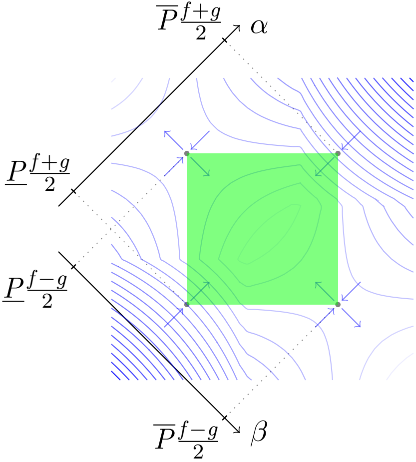
envelopes and a set



$$\underline{P}\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big)$$
$$=\min_{P\in\mathcal{M}\underline{P}} P\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big)$$

$$\Big\{\min_{\alpha\in\mathbb{R}}\max_{\beta\in\mathbb{R}} P\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big)\ \Big|\ P\in\mathcal{M}\underline{P}\Big\}$$

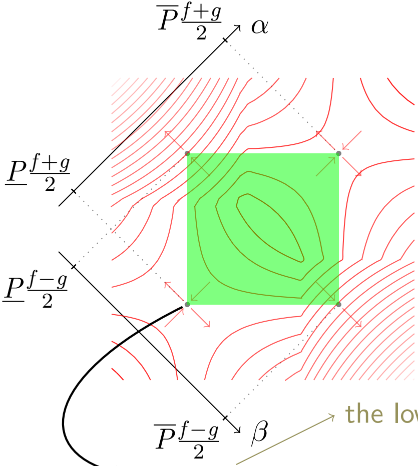
envelopes and a set



$$\overline{P}\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big)$$
$$=\max_{P\in\mathcal{M}\underline{P}} P\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big)$$

$$\Big\{\min_{\alpha\in\mathbb{R}}\max_{\beta\in\mathbb{R}} P\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big)\ \Big|\ P\in\mathcal{M}\underline{P}\Big\}$$

Lower & upper covariance notation



$$\underline{P}\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big) = \min_{P\in\mathcal{M}\underline{P}} P\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big)$$

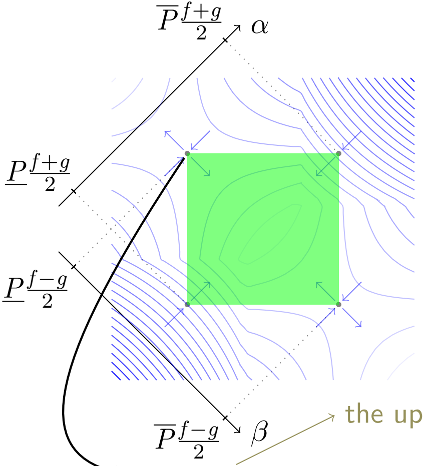
$$\Big\{ C_P\{f,g\} \;\Big|\; P\in\mathcal{M}\underline{P} \Big\}$$

the lower covariance of f and g under $\underline{P}$

$$\underline{C}_{\underline{P}}\{f,g\} := \min_{\alpha\in\mathbb{R}} \max_{\beta\in\mathbb{R}} \underline{P}\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big)$$

?

Lower & upper covariance notation



$$\overline{P}\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big) = \max_{P\in\mathcal{M}\underline{P}} P\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big)$$

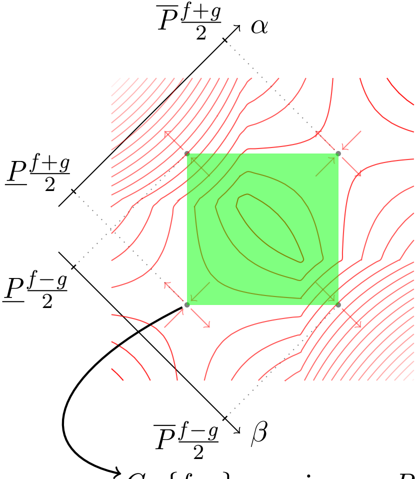
$$\Big\{ C_P\{f,g\} \;\Big|\; P\in\mathcal{M}\underline{P} \Big\}$$

the upper covariance of f and g under $\underline{P}$

$$\overline{C}_{\underline{P}}\{f,g\} := \max_{\beta\in\mathbb{R}}\min_{\alpha\in\mathbb{R}} \overline{P}\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big)$$

?

Lower & upper covariance



$$\underline{P}\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big) = \min_{P\in\mathcal{M}\underline{P}} P\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big)$$

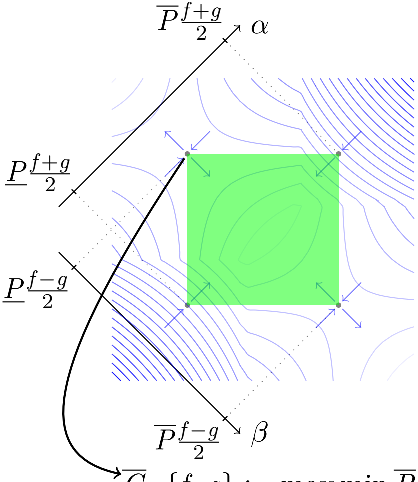
$$\left\{ C_P\{f,g\} \,\middle|\, P\in\mathcal{M}\underline{P} \right\}$$

the covariance envelope theorem

$$\underline{C}_{\underline{P}}\{f,g\} \coloneqq \min_{\alpha\in\mathbb{R}} \max_{\beta\in\mathbb{R}} \underline{P}\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big)$$

$$= \min\nolimits_{P\in\mathcal{M}\underline{P}} C_P\{f,g\}$$

Lower & upper covariance



$$\overline{P}\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big) = \max_{P\in\mathcal{M}\underline{P}} P\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big)$$

$$\Big\{ C_P\{f,g\} \;\Big|\; P\in\mathcal{M}\underline{P} \Big\}$$

the covariance envelope theorem

$$\overline{C}_{\underline{P}}\{f,g\} := \max_{\beta\in\mathbb{R}} \min_{\alpha\in\mathbb{R}} \overline{P}\big((\tfrac{f+g}{2}-\alpha)^2-(\tfrac{f-g}{2}-\beta)^2\big)$$

$$= \max\nolimits_{P\in\mathcal{M}\underline{P}} C_P\{f,g\}$$

Conclusion

We have found a definition of lower and upper covariance under coherent lower previsions that

- is direct, in the sense that it does not make use of the credal set of the lower prevision;
- and satisfies a covariance envelope theorem.

Moreover, it generalizes – as it should – the existing optimization problem definitions for covariance and (lower and upper) variance

Open questions

- Can this idea be extended to other, higher order central moments? In other words, can a definition be found for lower and upper versions of these moments under a coherent lower prevision that
  - is direct, in the sense that it does not make use of the credal set of the lower prevision;
  - and satisfies a higher order central moment envelope theorem?
- What is the (behavioral) meaning of an upper and lower covariance or, for that matter, lower and upper variance?