

Taking lower probabilities to their extreme

Erik Quaeghebeur

June 19, 2006

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Give maximum buying and minimum selling price
to cost-functions/gambles.

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N.B.: All these criteria can be expressed as linear constraints on \underline{P} .

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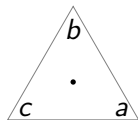
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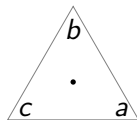
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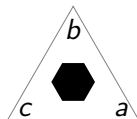
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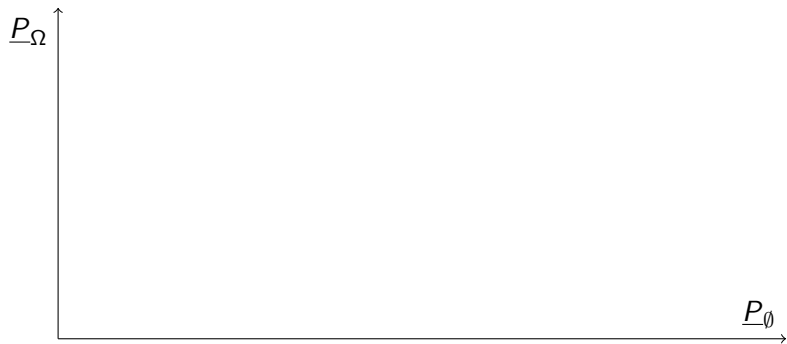


Coherent lower probabilities As a convex subset of the $(|\Omega| - 1)$ -dimensional unit simplex.



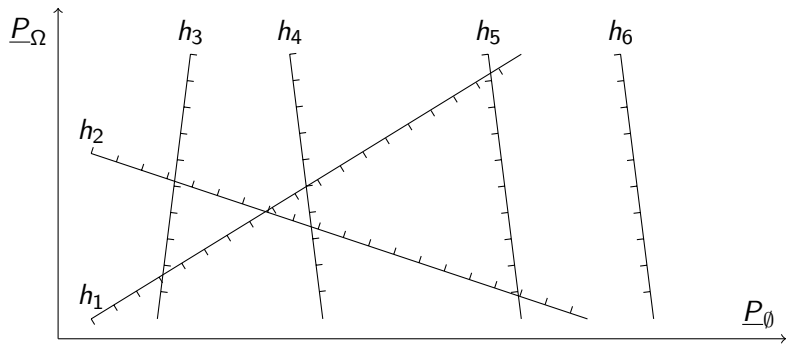
Constraints & vertices: a toy example

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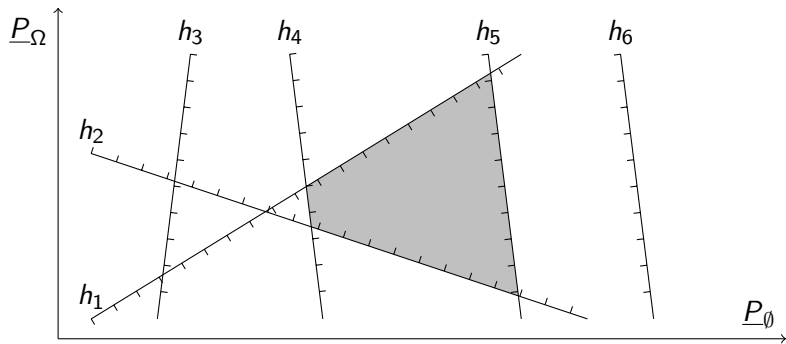
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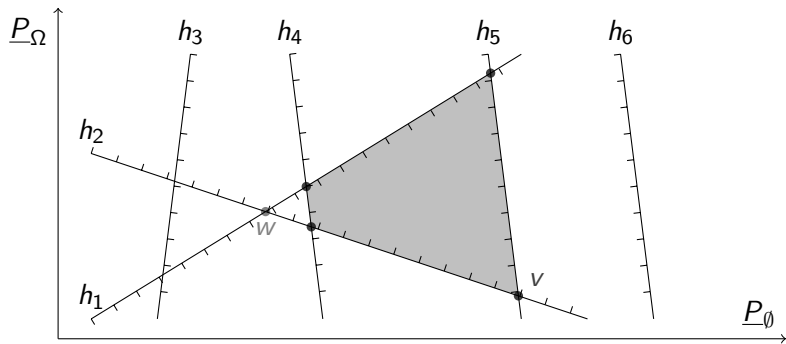
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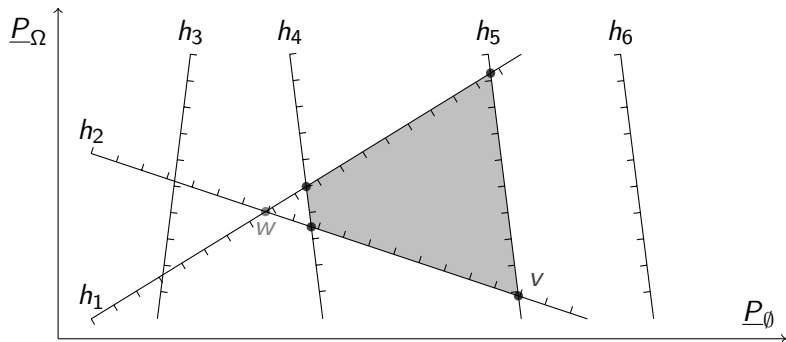


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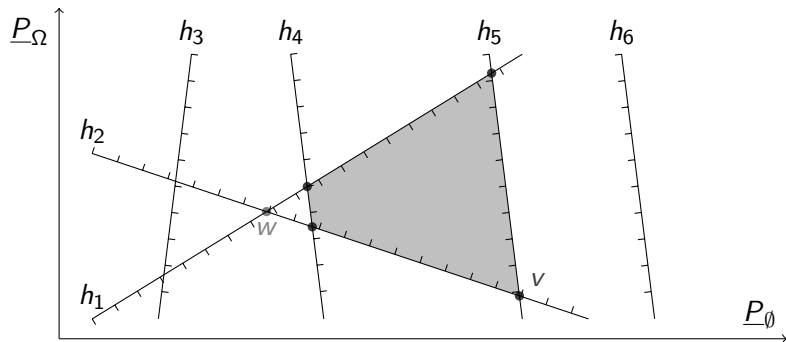


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Vertex enumeration From constraints to vertices: computationally heavy for large possibility spaces ($|\Omega| > 4$).

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- ▶ Make sure the supremum is attained everywhere.
- ▶ Consider only linearly independent sets of B 's or A and B 's for which $\bigcup_B B = \Omega$ or $\bigcup_B B = A$.

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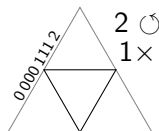
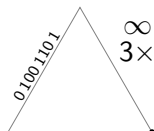
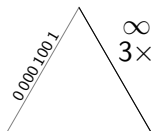
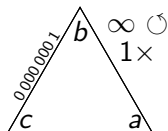
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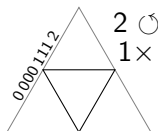
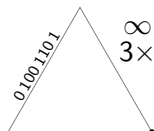
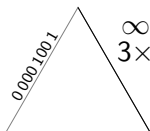
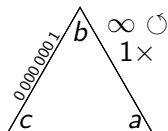
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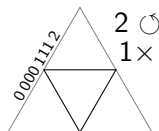
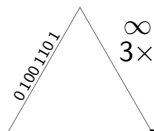
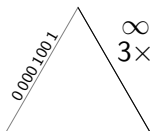
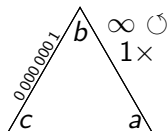
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- ▶ We're the first to find the 402 for $|\Omega| = 4$.
- ▶ For $|\Omega| = 5$ there are at least 1 743 093.

Conclusions

- ▶ Lower probabilities can be used to model uncertainty.
- ▶ We assembled and successfully used a method to compute extreme lower probabilities.

- ▶ Questions:
 - ▶ More efficient (vertex enumeration) algorithms?
 - ▶ Direct calculation possible?
 - ▶ What about extreme lower previsions?