Imprecise probability models for inference in exponential families

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Overview

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2. Specifying the details
3. A useful result
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5. History: how this research got started
6. An application: classification
7. Conclusions

The general idea

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- Obtain the corresponding *linear previsions* P_{C} and P_{P}.
- Imprecision: take a set of priors, use the lower envelope theorem to obtain coherent *lower previsions* $\underline{P}_{\mathsf{C}}$ and $\underline{P}_{\mathsf{P}}$.

Specifying the details: sampling model

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- *Exponential family sampling model* $\mathsf{Ef}(x \mid \psi)$: likelihood $\mathsf{L}_x(\psi)$, sufficient statistic $\tau(x)$ of fixed dimension.

$$\mathsf{Ef}(x \mid \psi) = \mathsf{a}(x) \exp(\langle \psi, \tau(x) \rangle - \mathsf{b}(\psi)).$$

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Multinomial sampling Likelihood function is a *multivariate Bernoulli* $\mathsf{Br}(x \mid \theta)$:

$$x \in \{0,1\}^d : \|x\| \leq 1; \quad \tau(x) = x;$$

$$\theta \in (0,1)^d : \|\theta\| < 1,\ \theta_0 = 1 - \sum_i \theta_i; \quad \psi(\theta) = \left(\ln(\frac{\theta_i}{\theta_0})\right)_{i=1}^{d};$$

$$\mathsf{a} = 1; \quad \mathsf{b}(\psi(\theta)) = \ln(\theta_0).$$

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Normal sampling Likelihood is a *Normal* $\mathsf{N}(x \mid \mu, \lambda)$:

$$x \in \mathbb{R}; \quad \tau(x) = (x, x^2);$$

$$\mu \in \mathbb{R},\ \lambda \in \mathbb{R}^+,\ \sigma^2 = \frac{1}{\lambda}; \quad \psi(\lambda, \mu) = (\lambda\mu, -\frac{1}{2}\lambda);$$

$$\mathsf{a} = \frac{1}{\sqrt{2\pi}}; \quad \mathsf{b}(\psi(\mu, \lambda)) = \frac{\lambda\mu^2 - \ln(\lambda)}{2}.$$

Specifying the details: conjugate

- Choose some *prior* $C(\psi)$: obtain a *posterior* after observing samples.

Specifying the details: conjugate

- Choose a *conjugate prior* $\mathsf{CEf}(\psi \,|\, n^0, y^0)$: easily obtain a *posterior* $\mathsf{CEf}(\psi \,|\, n^k, y^k)$ after observing k samples.

$$\mathsf{CEf}(\psi \,|\, n, y) = \mathsf{c}(n, y) \exp(n \left[\langle \psi, y \rangle - \mathsf{b}(\psi)\right])$$

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Multinomial sampling The conjugate distribution is a *Dirichlet distribution* $\mathsf{Di}(\theta \,|\, ny, ny_0)$:

$$y \in (0,1)^d : \|y\| < 1,\; y_0 = 1 - \sum_i y_i;$$

$$\mathsf{c}(n, y) = \frac{\Gamma(n)}{\Gamma(ny_0) \prod_i \Gamma(ny_i)}.$$

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Normal sampling The conjugate distribution is a *Normal-gamma distribution*

$\mathsf{N}(\mu \,|\, y_1, n\lambda)\mathsf{Ga}(\lambda \,|\, \frac{n+3}{2}, \frac{n\left[y_2 - {y_1}^2\right]}{2})$:

$$y \in \mathbb{R} \times \mathbb{R}^+ : y_2 - {y_1}^2 > 0;$$

$$\mathsf{c}(n, y) = \frac{2\sqrt{n}}{\sqrt{2\pi}} \frac{\left[\frac{n\left[y_2 - {y_1}^2\right]}{2}\right]^{\frac{n+3}{2}}}{\Gamma(\frac{n+3}{2})}.$$

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$$\mathsf{PEf}(x \,|\, n, y) = \int_{\Psi} \mathsf{CEf}(\cdot \,|\, n, y)\mathsf{L}_x = \frac{\mathsf{c}(n, y)\mathsf{a}(x)}{\mathsf{c}(n+1, \frac{ny+\tau(x)}{n+1})}.$$

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Multinomial sampling The predictive distribution is a *Dirichlet-multinomial distribution* $\mathsf{DiMn}(x \,|\, ny, ny_0)$.

Normal sampling The predictive distribution is a *Student distribution* $\mathsf{St}(x \,|\, y_1, \frac{n+3}{n+1}\frac{1}{y_2-y_1^2}, n+3)$.

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$$P_{\mathsf{C}}(f \,|\, n^k, y) = \int_{\Psi} \mathsf{CEf}(\cdot \,|\, n^k, y) f, \quad f \in \mathcal{L}(\Psi) \approx [\Psi \to \mathbb{R}].$$

and

$$P_{\mathsf{P}}(f \,|\, n^k, y) = \int_{\mathcal{X}} \mathsf{PEf}(\cdot \,|\, n^k, y) f, \quad f \in \mathcal{L}(\mathcal{X}) \approx [\mathcal{X} \to \mathbb{R}].$$

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$$\underline{P}_{\mathsf{C}}(\cdot \,|\, n^k, \mathcal{Y}^k) = \inf_{y \in \mathcal{Y}^k} P_{\mathsf{C}}(\cdot \,|\, n^k, y).$$

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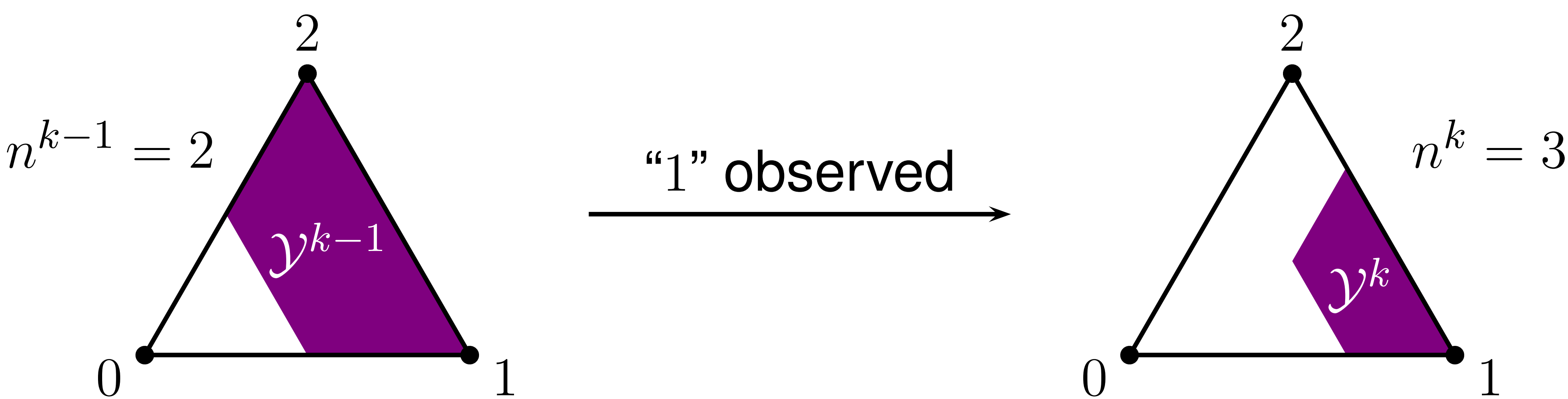
Multinomial sampling



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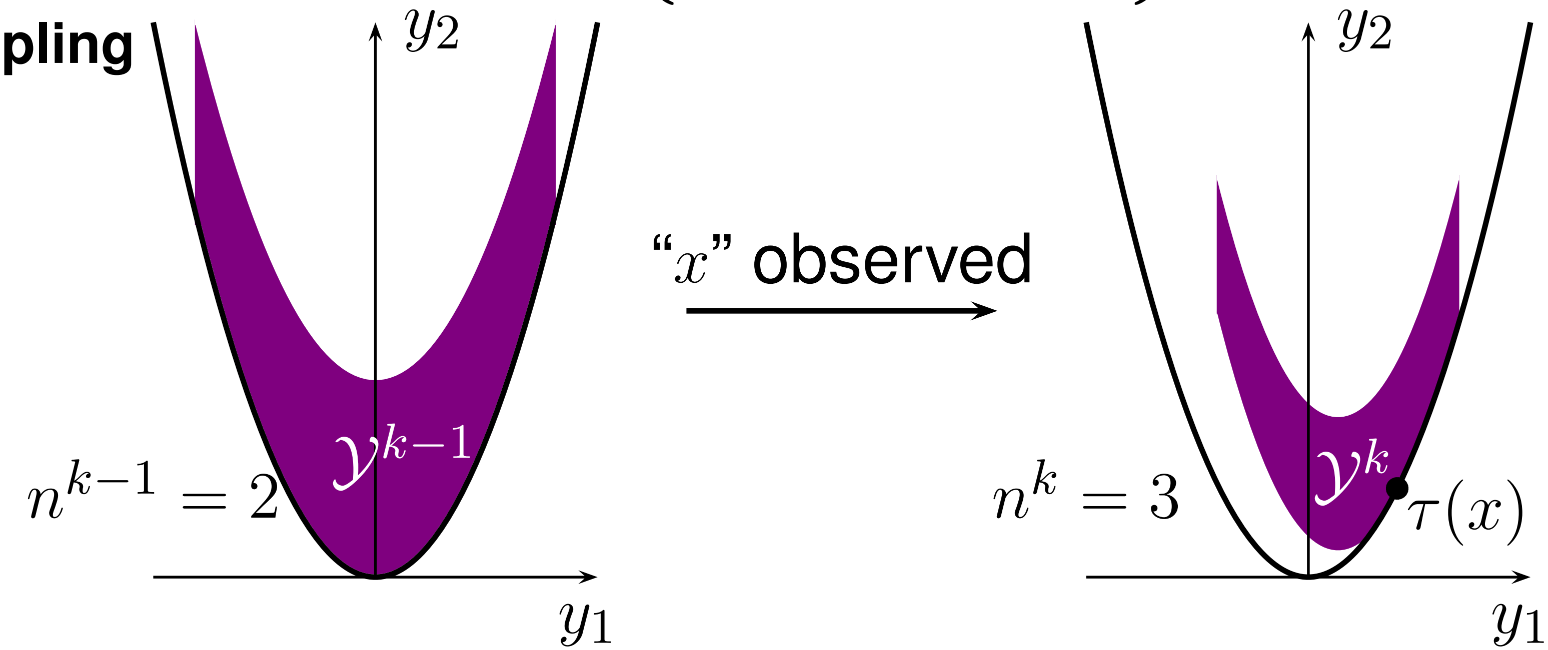

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- Using the IDM: problems with optimization problems.
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- ...the realization that the idea underlying the IDM for multinomial sampling generalizes to all exponential family sampling models:
  - common interpretation for parameters n and y;
  - easy updating.
- However, using these models: again possible problems with optimization problems.

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- Classically, both models are IDMM's; here, any $\underline{P}_{\mathsf{P}}(\cdot \mid n_{\mathcal{A}|\mathcal{C}}, \mathcal{Y}_{\mathcal{A}|\mathcal{C}})$ is possible.
- Advantages:
  - allows for continuous attributes;
  - straightforward training.
- Disadvantage: optimization problems are harder to solve.

An application: classification

- Example optimizaton problems:

Multinomial sampling (i.e., multiple discrete attributes)

$$c' \succ c'' \iff$$

$$\inf_{y \in \mathcal{Y}_C} \left[y_{c'} \prod_i \inf_{y_{\mathcal{A}_i|c'} \in \mathcal{Y}_{\mathcal{A}_i|c'}} y_{a_i|c'} - y_{c''} \prod_i \sup_{y_{\mathcal{A}_i|c''} \in \mathcal{Y}_{\mathcal{A}_i|c''}} y_{a_i|c''} \right] > 0.$$

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Normal sampling (i.e., one normal attribute) Replace the products above by the $\inf / \sup_{y_{\mathcal{A}|c} \in \mathcal{Y}_{\mathcal{A}|c}}$ of

$$\sqrt{\frac{n_{\mathcal{A}|c}}{n_{\mathcal{A}|c}+1}} \frac{\Gamma(\frac{n_{\mathcal{A}|c}+4}{2})}{\Gamma(\frac{n_{\mathcal{A}|c}+3}{2})} \frac{[n_{\mathcal{A}|c} y_{\mathcal{A}|c,2} - n_{\mathcal{A}|c} y^2_{\mathcal{A}|c,1}]^{\frac{n_{\mathcal{A}|c}+3}{2}}}{[n_{\mathcal{A}|c} y_{\mathcal{A}|c,2} + a^2 - \frac{1}{n_{\mathcal{A}|c}+1}[n_{\mathcal{A}|c} y_{\mathcal{A}|c,1} + a]^2]^{\frac{n_{\mathcal{A}|c}+4}{2}}}.$$

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  - one for making inferences about the parameter describing the sampling model;
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- Applicable for a large range of sampling models. . .
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- However, difficult optimization problems might severely limit their use.

Time for questions!

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