CONNECTING CHOICE FUNCTIONS AND SETS OF DESIRABLE GAMBLES

ARTHUR VAN CAMP, GERT DE COOMAN, AND ERIK QUAEGHEBEUR

ABSTRACT. We study Seidenfeld, Schervish, and Kadane's notion of choice functions, and want to make them accessible to people who are familiar with sets of desirable gambles. We relate both theories explicitly using their derived strict partial orderings. We give an expression for the most conservative extension of a set of desirable gambles to a choice function. Because it is important for inference purposes, we also make a link with belief structures.

1. INTRODUCTION

Our aim is to consider Seidenfeld et al.'s [2010] notion of a choice function, and to make it accessible to people who are familiar with sets of desirable gambles, or other concepts derived from them. The first step in doing so, is to connect both theories explicitly. Because choice functions are defined on sets of horse lotteries, while sets of desirable gambles rely on the notion of gambles—which do not translate naturally to horse lotteries—we search for a language to connect them. One thing both theories have in common, is that they use specific partial orders. Therefore, in Sections 2 and 4, we focus on these partial orders, which will enable us to relate both theories in a way that also allows ideas in the theory of sets of desirable gambles to be embedded into choice function theory. In particular, we give an expression for the most conservative extension of a set of desirable gambles to a choice function.

We also make a link with belief structures. In Section 3, we introduce an 'is not more informative than' relation on choice functions, under which these models form a belief structure. This is important, as it allows us to introduce notions of conservative inference (natural extension) into choice function theory.

2. CHOICE FUNCTIONS DEFINED ON SETS OF HORSE LOTTERIES

As a basic tool to model an agent's uncertainty about some variable X that takes values in a possibility space $\mathcal{X}$, we use *choice functions* defined on sets of *horse lotteries*. They are—in contradistinction with sets of desirable gambles [cf., e.g., De Cooman and Quaeghebeur, 2012, Quaeghebeur, 2014, Couso and Moral, 2011, Walley, 2000]—capable of modelling beliefs corresponding to a non-convex set of probability measures.

2.1. **Horse lotteries.** Consider a countable set $\mathbb{K}$ of rewards. A *simple horse lottery* on $\mathbb{K}$ is a probability mass function p on $\mathbb{K}$, *i.e.*, a map $p : \mathbb{K} \to \mathbb{R}$ such that $p(r) \geq 0$ for all r in $\mathbb{K}$ and $\sum_{r \in \mathbb{K}} p(r) = 1$. A *horse lottery* is a map $H : \mathcal{X} \times \mathbb{K} \to \mathbb{R}$ such that for each x in $\mathcal{X}$, the partial map $H(x, \cdot)$ is simple horse lottery on $\mathbb{K}$, the agent's 'random' reward. The set of all horse lotteries on $\mathcal{X}$ with reward set $\mathbb{K}$ is denoted by $\mathcal{H}_{\mathbb{K}}$.

Let $\{\top, \bot\} \subseteq \mathbb{K}$. For two horse lotteries H_1 and H_2 in $\mathcal{H}_{\mathbb{K}}$ such that $H_1(\cdot, r) = H_2(\cdot, r) = 0$ when $r \notin \{\top, \bot\}$, we say that H_1 *weakly dominates* H_2—and we write $H_1 \trianglerighteq H_2$—whenever $H_1(x, \top) \geq H_2(x, \top)$ for all x in $\mathcal{X}$. So $\top$ is the reward that consist in winning a prize that you value, and $\bot$ consist in not winning that prize.

In Section 4, we will consider the particular case $\mathbb{K} = \{\top, \bot\}$, and we then denote the corresponding set of horse lotteries by $\mathcal{H}$.

2.2. **Choice functions and relations on sets of horse lotteries.** We define a choice function as a map

$$C\colon \mathcal{Q}(\mathcal{H}_{\mathbb{K}}) \to \mathcal{Q}(\mathcal{H}_{\mathbb{K}}) \cup \{\emptyset\} : O \mapsto C(O) \quad \text{such that} \quad C(O) \subseteq O.$$

This is the same definition as in Sen [1977], Aizerman [1984] and Seidenfeld et al. [2010, Section 1], except that we restrict attention to finite *option sets* of horse lotteries. So $\mathcal{Q}(\mathcal{H}_{\mathbb{K}})$ denotes the collection of all non-empty finite subsets of $\mathcal{H}_{\mathbb{K}}$. The idea is that an agent's choice function selects a preferred subset of horse lotteries from an option set O, according to the agent's beliefs.

As an alternative to a choice function C, we can also consider the binary relation—called a *choice relation*—$<_C$ on $\mathcal{Q}(\mathcal{H}_{\mathbb{K}})$ as in Kadane et al. [2004, Section 2], defined for all O_1 and O_2 in $\mathcal{Q}(\mathcal{H}_{\mathbb{K}})$ by

$$O_1 <_C O_2 \quad \text{if and only if} \quad C(O_1 \cup O_2) \subseteq O_2 \setminus O_1.$$

The set of all choice relations is denoted by $\mathcal{C}$. We interpret $O_1 <_C O_2$ as the agent choosing option set O_2 over option set O_1: when presented with $O_1 \cup O_2$, she will accept a subset O_2 outside of O_1. The following technical result can be obtained:

Proposition 1. *Given a choice function C, the associated choice relation $<_C$ satisfies*

(i) *if $O_2 \setminus O_1 \subseteq O \subseteq O_1 \cup O_2$, then $O_1 <_C O_2 \Leftrightarrow O_1 <_C O$,*
(ii) *if $O_1 \cup O_2 \subseteq O$, then $(O_1 <_C O$ and $O_2 <_C O) \Leftrightarrow O_1 \cup O_2 <_C O$,*

for all O, O_1, and O_2 in $\mathcal{Q}(\mathcal{H}_{\mathbb{K}})$.

Any relation satisfying properties (i) and (ii) will from here on be called a *choice relation*. Conversely, given a choice relation $<$, we can define the associated choice function $C_<$ for all O in $\mathcal{Q}(\mathcal{H}_{\mathbb{K}})$ by

$$C_<(O) := \bigcap \{O' \subseteq O : O' \nless O\}.$$

Given these definitions, we can move between both representations:

Proposition 2. *Given a choice function C, we have $C_{<_C} = C$; and given a choice relation $<$, we have $<_{C_<} = <$.*

Because it will turn out useful for the connection with sets of desirable gambles, we choose to develop everything in terms of choice relations.

2.3. **Coherence.** Seidenfeld et al. [2010, Section 3] call a choice function C coherent if there is a non-empty set of probability-utility pairs $\mathcal{S}$ such that $C(O)$ is the set of options in O that maximise expected utility for some probability-utility pair in $\mathcal{S}$. They also provide an axiomatisation for this type of coherence, based on the one for binary preferences in Anscombe and Aumann [1963]. One of their axioms is an 'Archimedean' continuity condition.

We prefer to define coherence in terms of axioms directly, without reference to probabilities and utilities. In such a context, we see no compelling reason to adopt an Archimedean axiom. Therefore, we use Seidenfeld et al.'s [2010] axioms, omitting their Archimedean one.

Axioms (coherent choice relations)**.** We call a choice relation $<$ on $\mathcal{Q}(\mathcal{H}_{\mathbb{K}})$ coherent if

$<_1$. $O \nless O$, so $<$ is irreflexive;
$<_2$. $\{H_\bot\} < \{H_\top\}$, where H_r with r in $\mathbb{K}$ is a horse lottery such that $H_r(\cdot, r) = 1$;

$<_3$. (a) if $O_1 < O_2$ and $O_1 \subseteq O_2 \subseteq O$, then $O_1 < O$;
(b) if $O_1 < O_2$ and $O \subseteq O_1 \subseteq O_2$, then $O_1 \setminus O < O_2 \setminus O$;

$<_4$. (a) $O_1 < O_2$ if and only if for all α in $(0,1]$

$$\{\alpha G + (1-\alpha)H : G \in O_1\} < \{\alpha G + (1-\alpha)H : G \in O_2\};$$

(b) if $H \in O$, $O \subseteq O_1 \subseteq \mathrm{co}(O)$—the convex hull of O—and $\{H\} < O_1$, then $\{H\} < O$;

$<_5$. if H_2 weakly dominates H_1 ($H_2 \trianglerighteq H_1$) and for all G in $O \setminus \{H_1, H_2\}$, we have:
(a) if $H_2 \in O$ and $\{G\} < O \cup \{H_1\}$, then $\{G\} < O$;
(b) if $H_1 \in O$ and $\{G\} < O$, then $\{G\} < \{H_2\} \cup O \setminus \{H_1\}$;

for all O, O_1 and O_2 in $\mathcal{Q}(\mathcal{H}_{\mathbb{K}})$, and H, H_1 and H_2 in $\mathcal{H}_{\mathbb{K}}$.

The coherent choice relations are collected in $\mathcal{C}_{\mathrm{coh}}$.

Proposition 3 (Cf. Kadane et al., 2004, Lemma 1)**.** *A coherent choice relation $<$ is a strict partial order: it is transitive, so $(O_1 < O_2$ and $O_2 < O_3)$ implies $O_1 < O_3$ for all O_1, O_2, O_3 in $\mathcal{Q}(\mathcal{H}_{\mathbb{K}})$, as well as irreflexive.*

3. Ordering of choice relations

Given two choice relations $<_1$ and $<_2$, we call $<_1$ *not more informative than* $<_2$ if $<_1 \subseteq <_2$. The set $(\mathcal{C}, \subseteq)$ of choice relations is partially ordered by the set inclusion of their graphs. Following De Cooman [2005], we introduce infimum and supremum operators—or meets and joins—for collections of choice relations $\boldsymbol{<} \subseteq \mathcal{C}$ as follows:

$$\inf \boldsymbol{<} = \bigcap_{< \in \boldsymbol{<}} <, \qquad \sup \boldsymbol{<} = \bigcup_{< \in \boldsymbol{<}} <.$$

For inference purposes, an important subclass of general belief models are those that form a *belief structure* [cf. De Cooman, 2005].

Proposition 4. *The structure $(\mathcal{C}, \mathcal{C}_{coh}, \subseteq)$ is a belief structure:*

(i) *The partially ordered set $(\mathcal{C}, \subseteq)$ is a complete lattice: for any subset $\boldsymbol{<}$ of $\mathcal{C}$, its supremum* $\sup \boldsymbol{<}$ *and its infimum* $\inf \boldsymbol{<}$ *with respect to the order $\subseteq$ exists.*
(ii) *The partially ordered set $(\mathcal{C}_{coh}, \subseteq)$ is an* intersection structure*, meaning that $\mathcal{C}_{coh}$ is closed under arbitrary non-empty infima: for any non-empty subset $\boldsymbol{<}$ of $\mathcal{C}_{coh}$,* $\inf \boldsymbol{<} \in \mathcal{C}_{coh}$.
(iii) *$(\mathcal{C}_{coh}, \subseteq)$ does not have a top.*

This is interesting because it guarantees that we can use a closure operator to check consistency and coherence [De Cooman, 2005, Propositions 1 and 2].

4. Connecting choice functions and sets of desirable gambles

In this section, we are looking for a connection between coherent choice functions with reward set $\mathbb{K} = \{\top, \bot\}$ and coherent sets of desirable gambles. Coherent sets of desirable gambles correspond to a strict partial order of *elements* (namely, gambles), rather than a strict partial order of *sets* (of horse lotteries) as is the case for choice functions. Therefore, the language of choice functions has a greater flexibility in expressing beliefs, so we expect a *many to one*-relation with sets of desirable gambles: with a choice function there will correspond a unique set of desirable gambles; but conversely, given a set of desirable gambles, there may be multiple compatible choice functions.

4.1. **Gambles.** A *gamble* f is a bounded real-valued function on the possibility space $\mathcal{X}$. It is interpreted as an uncertain reward $f(X)$. If the value of X turns out to be x in $\mathcal{X}$, then f results in a payoff $f(x)$. The set of all gambles on $\mathcal{X}$ will be denoted by $\mathcal{G}(\mathcal{X})$. As an example, for any horse lottery H in $\mathcal{H}$, $H(\cdot,\top)$ is a gamble.

4.2. **Sets of desirable gambles and partial order on gambles.** Some gambles can be preferred to others by the agent, depending on her beliefs about X. We call all gambles that the agent prefers to 0 *desirable*, and we collect them in her *set of desirable gambles* $\mathcal{D}$.

Based on an agent's statement of $\mathcal{D}$, we can define a strict partial order—called a *(strict partial) preference relation*—$\prec_{\mathcal{D}}$ on $\mathcal{G}(\mathcal{X})$ as in Quaeghebeur et al. [2014, Section 3] and Quaeghebeur [2014], defined for all f and g in $\mathcal{G}(\mathcal{X})$ by

$$f \prec_{\mathcal{D}} g \quad \text{if and only if} \quad g - f \in \mathcal{D}.$$

Conversely, given a preference relation $\prec$ on gambles, we can define the corresponding set of desirable gambles $\mathcal{D}_{\prec}$ as

$$\mathcal{D}_{\prec} \coloneqq \{f \in \mathcal{G}(\mathcal{X}) : 0 \prec f\}.$$

As is the case for choice functions and choice relations, we have a one-to-one correspondence between sets of desirable gambles and preference relations.

4.3. **Coherence.** As is the case for choice functions, in order for a set of desirable gambles to be *coherent*, it needs to satisfy some rationality criteria. Again we state these using the corresponding partial orders, i.e., in terms of preference relations.

Axioms (coherent preference relations)**.** We call a preference relation $\prec$ coherent if

$\prec_1$. $f \not\prec f$;
$\prec_2$. if $\inf f > 0$, then $0 \prec f$;*
$\prec_3$. $f \prec g \Leftrightarrow \mu f + (1-\mu)h \prec \mu g + (1-\mu)h$;
$\prec_4$. if $f \prec g$ and $g \prec h$, then $f \prec h$;

for all f, g and h in $\mathcal{G}(\mathcal{X})$ and $0 < \mu \leq 1$.

4.4. **Connection between choice functions and sets of desirable gambles.** With a coherent choice relation $<$ on $\mathcal{Q}(\mathcal{H})$, we associate a preference relation $\prec$ on $\mathcal{G}(\mathcal{X})$:

$$f \prec g \quad \text{if and only if}$$
$$(\exists H_1, H_2 \in \mathcal{H})\big(\{H_1\} < \{H_2\} \quad \text{and} \quad (\exists \alpha \in \mathbb{R}_{>0})\alpha(f-g) = H_1(\cdot,\top) - H_2(\cdot,\top)\big) \qquad (\star)$$

for all f and g in $\mathcal{G}(\mathcal{X})$.

Proposition 5. *The preference relation as defined in Eq.* $(\star)$ *is coherent.*

Conversely, a coherent preference relation $\prec$ can be coherently 'lifted' to multiple choice relations $<$ that satisfy Eq. $(\star)$, but we are looking for the single one that is most conservative—such that $O_1 < O_2$ is satisfied for the least number of option sets O_1 and O_2.

Proposition 6. *Given a coherent preference relation $\prec$, the most conservative coherent choice relation $<$ on $\mathcal{Q}(\mathcal{H})$ that satisfies Eq.* $(\star)$ *is given by*

$$O_1 < O_2 \Leftrightarrow (\forall H_1 \in O_1)(\exists H_2 \in O_2) H_1(\cdot,\top) \prec H_2(\cdot,\top)$$

for all O_1 and O_2 in $\mathcal{Q}(\mathcal{H})$.

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* Often 'if $f > 0$, then $f \in \mathcal{D}$' is used. The weaker form we use allows for a more direct correspondence with choice relations.

5. CONCLUSIONS

We have bridged choice function theory and the theory of sets of desirable gambles. We were motivated by the fact that the latter language is incapable of representing beliefs corresponding to non-convex sets of probabilities, while choice functions are. Hence, given a set of desirable gambles, there may be multiple corresponding choice functions, and we found an expression for the most conservative amongst them.

We are interested in extending this work to the case of non-finite option sets.

We furthermore plan to extend our investigation of the structural properties of the belief structure $(\mathcal{C}, \mathcal{C}_{\mathrm{coh}}, \subseteq)$, initiated in Section 3. For example, the question whether this is a *strong* belief structure—meaning that every coherent choice function is the infimum of its dominating maximal coherent choice functions—deserves more attention, as does the notion of natural extension for choice functions.

Also, it might be interesting to find a bridge between choice functions and the accept and reject statement-based models, introduced in Quaeghebeur et al. [2014], which are more expressive than sets of desirable gambles.

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GHENT UNIVERSITY, SYSTEMS RESEARCH GROUP, TECHNOLOGIEPARK–ZWIJNAARDE 914, 9052 ZWIJNAARDE, BELGIUM
E-mail address: {arthur.vancamp,gert.decooman}@ugent.be

CENTRUM WISKUNDE & INFORMATICA, SCIENCE PARK 123, 1098 XG AMSTERDAM, THE NETHERLANDS
E-mail address: Erik.Quaeghebeur@cwi.nl